

# Euclidean TSP Between Two Nested Convex Obstacles

August 11, 2005

**Jeff Abrahamson**, Dept. of Computer Science, Drexel University, Philadelphia, PA, USA,  
jeffa@cs.drexel.edu.

**Ali Shokoufandeh**, Dept. of Computer Science, Drexel University, Philadelphia, PA, USA,  
ashokouf@cs.drexel.edu.

**Pawel Winter**, Dept. of Computer Science, University of Copenhagen, Denmark,  
pawel@diku.dk.

This paper was published in Information Processing Letters, (95) 370–375, 2005.
---

## Abstract

We present a polynomial-time algorithm for a variant of the Euclidean traveling salesman tour where  $n$  vertices are on the boundary of a convex polygon  $P$  and  $m$  vertices form the boundary of a convex polygonal obstacle  $Q$  completely contained within  $P$ . In the worst case the algorithm needs  $O(m^2 \lg m + m^2 n)$  time and  $O(nm + m^2)$  space.

Keywords: Computational Geometry; Traveling Salesman Problem; Convex Analysis; Network Flow

## 1 Introduction

The Euclidean TSP (ETSP) is the problem of finding a tour of minimum length through a given set of points in  $d$ -dimensional Euclidean space. In this paper, we will address a variant of the ETSP in which the points are the vertices of a convex polygon  $P$  and a convex obstacle  $Q$  located completely in the interior of  $P$ . We give a polynomial time algorithm for finding the shortest tour  $T$  through the vertices of  $P$  and  $Q$  while completely avoiding the interior of  $Q$ .

Deineko et al. [2] considered a related variant of the ETSP with a convex polygon  $P$  and a set of points on a line segment  $Q$  inside  $P$ . They referred to this problem as the *convex-hull-and-line* ETSP, and gave an  $O(m^2 + mn)$  time and  $O(m + n)$  space algorithm, where  $n$  and  $m$  are the number of  $P$ - and  $Q$ -vertices, respectively. Cutler [1] gave an  $O(n^3)$  time and  $O(n^2)$  space dynamic programming algorithm for solving the *3-line* ETSP where all points lie on three distinct parallel lines in the plane. Rote [5] extended this result to  $m$ -line ETSP by giving a polynomial dynamic programming algorithm for a fixed number of lines  $m$ .

It would appear that for an optimal solution to the problem of finding a tour through the vertices of a convex polygon  $P$  and those of the convex polygonal obstacle  $Q$  contained within  $P$ , the tour must visit each vertex exactly once. It is not hard to see, however, that some tours that visit some vertices of  $Q$  more than once may be shorter than tours that visit each  $Q$  vertex precisely once. Further, optimal solutions can be characterized as consisting of detours to  $Q$ -vertices between appropriately chosen pairs of consecutive  $P$ -vertices. Based on this, and by transforming the problem to  $O(m)$  shortest paths computations in an appropriately defined digraph, we give an  $O(m^2 \lg m + mn)$  time and  $O(m^2 + mn)$  space algorithm, where  $m$  and  $n$  represent the number of vertices in  $P$  and  $Q$  respectively. Our approach to this problem is a generalization of that used in [2] to solve the convex-hull-and-line ETSP.

## 2 Basic Definitions

In the following we assume a distance function that obeys the triangle inequality and is additive:  $d(a, c) = d(a, b) + d(b, c)$  for all co-linear points  $a$ ,  $b$ , and  $c$  with  $b$  between  $a$  and  $c$ . We often implicitly use the following seemingly intuitive result:

**Proposition 1** *Let  $abc$  be a triangle and  $C$  a convex chain in its interior between  $a$  and  $c$  of length  $\|C\|$ . Then  $d(a, b) + d(b, c) > \|C\|$ .*

**Sketch of Proof:** The proof is by repeated application of the triangle inequality. Suppose  $x$  is some point in the interior of  $abc$  such that  $C$  is in the interior of the triangle  $axc$ . Clearly showing that  $d(a, x) + d(x, c) < d(a, b) + d(b, c)$  is sufficient to prove the proposition. Extend the line segment  $ax$  until it bisects edge  $bc$  and call the point of intersection  $f$ . Then we have that  $ab + bc > af + fc > ax + xc$ .  $\square$

Let  $P$  denote a convex polygon in the Euclidean plane and  $Q$  a convex obstacle that is defined to be completely in the interior of  $P$ , both in general position (taken together). Let  $p_1, \dots, p_n$  denote the vertices of  $P$  and  $q_1, \dots, q_m$  be the vertices of  $Q$ ,  $m \geq 3$ . Throughout this paper we will assume that vertices are numbered in the clockwise direction and that subscripts are interpreted modulo  $n$  for  $P$  and  $m$  for  $Q$ . We will also denote two consecutive  $P$ - or  $Q$ -vertices as  $v$  and  $v^+$ .

An *edge*  $uv$  that connects a pair of points  $u$  and  $v$  is a straight line segment between  $u$  and  $v$  of length  $\|uv\|$ . If the edge between any points  $u$  and  $v$  avoids the obstacle  $Q$ , then the two points are said to be *visible* to each other. A *path* is a sequence  $u_1, u_2, \dots, u_k$  of vertices and the interconnecting edges  $u_1u_2, u_2u_3, \dots, u_{k-1}u_k$ . We will often write the path  $\pi = u_1 \rightsquigarrow u_k = u_1u_2 \oplus u_2u_3 \oplus \dots \oplus u_{k-1}u_k$ , where  $\oplus$  denotes edge concatenation.

Consider two paths  $\pi_1$  and  $\pi_2$  that share a common vertex  $v$ . We say that  $v$  is a *vertex intersection* if the two paths cross each other at  $v$ . If they touch at  $v$  but do not cross, we call  $v$  a *touch point*. If some edges  $e_1 \in \pi_1$  and  $e_2 \in \pi_2$  intersect, we say that  $\pi_1$  and  $\pi_2$  have an *edge intersection*. Two paths intersect if they have either a vertex or an edge intersection.

If  $\pi = u_1 \rightsquigarrow u_k$  is a path and  $u_1 = u_k$ , then the path is called a *tour*. A *simple* tour  $T$  has no duplicate vertices except the necessary first and last. A tour  $T$  is *weakly-simple* if it has no intersections except possibly for backtracking ( $\dots \oplus vv' \oplus v'v \oplus \dots$ , see figure 1). Thus, both simple and weakly-simple tours have well-defined interiors. Finally, in the context of the problem addressed in this paper, a weakly-simple tour with no  $P$ -vertices is said to be *degenerate*.

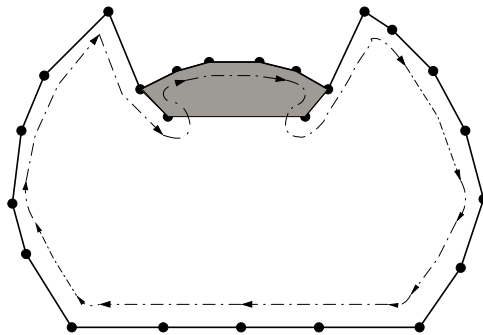


Figure 1: A shortest tour (bold path) that would be longer if backtracking were not permitted.

A tour through all vertices of  $P$  and  $Q$  involves three types of edges: *polygon edges* connecting  $P$ -vertices, *obstacle edges* connecting consecutive  $Q$ -vertices, and *cross-over edges* connecting  $P$ -vertices with  $Q$ -vertices or non-consecutive vertices of  $P$ . We define  $\|q_i \rightsquigarrow q_j\|$  to be the length of the polygonal path  $q_i \rightsquigarrow q_j = q_i q_{i+1} \oplus \dots \oplus q_{j-1} q_j$ . Note that the path  $q_i \rightsquigarrow q_j$  is clockwise, and that  $(q_i \rightsquigarrow q_j) \oplus (q_j \rightsquigarrow q_i) = Q$ .

Consider a polygon edge  $p_k p_k^+$ . We define a clockwise *detour*  $d_{i,j}^k$  of  $p_k p_k^+$ , for any pair of (not necessarily distinct)  $Q$ -vertices  $q_i$  and  $q_j$  to be the path  $d_{i,j}^k = P_Q(p_k, q_i) \oplus (q_i \rightsquigarrow q_j) \oplus P_Q(q_j, p_k^+)$ , where  $P_Q(u, v)$  denotes a shortest obstacle-avoiding path from  $u$  to  $v$ . Note that the path  $P_Q(u, v)$  may traverse points of  $Q$  and even result, when added to  $q_i \rightsquigarrow q_j$ , in retracing vertices. We shall refer to such duplicate traversals in  $P_Q(\cdot, \cdot)$  as *backtracking*. Two detours are *disjoint* if the sets of their  $Q$ -vertices are disjoint. The *incremental cost*  $c_{i,j}^k$  of the detour  $d_{i,j}^k$  is

$$c_{i,j}^k = \|P_Q(p_k, q_i)\| + \|q_i \rightsquigarrow q_j\| + \|P_Q(q_j, p_k^+)\| - \|p_k p_k^+\|.$$

Let  $d_{i,j}$  denote a cheapest clockwise detour through  $q_i \rightsquigarrow q_j$  taken over all polygon edges  $p_k p_k^+$ . Set  $d_{i,j} = \arg \min\{d_{i,j}^1, d_{i,j}^2, \dots, d_{i,j}^n\}$  and let  $c_{i,j} = \min\{c_{i,j}^1, c_{i,j}^2, \dots, c_{i,j}^n\}$ . That is,  $d_{i,j}$  is a detour of cost  $c_{i,j}$ .

### 3 Structural Properties

In this section we provide a structural characterization of a shortest tour  $T$  through the vertices of  $P$  and  $Q$ . It will facilitate a transformation of the original problem to  $m$  shortest paths problems in an appropriately

defined digraph  $G$ .

**Lemma 2** *An optimal tour  $T$  has intersections only on  $Q$  vertices.*

**Proof:** Assume to the contrary that the optimal tour  $T$  contains an intersection on a vertex of  $P$  or in the region between  $P$  and  $Q$ . (The region outside  $P$  is prohibited by convexity and the region inside  $Q$  is prohibited by construction.) Of the four vertices that form an intersection, at least two must be vertices of  $P$ , since  $Q$  is convex and no three of its vertices are co-linear. Therefore, an intersection in  $T$  can only occur in the following four cases:

1. **All four vertices are on  $P$ .** Let  $p_1, p_2, p_3$  and  $p_4$  denote the four vertices and suppose that  $p_1p_3$  and  $p_2p_4$  intersect in some point  $v$ , see Figure 2. The segments determined by  $p_1p_3$  and  $p_2p_4$  divide the quadrilateral  $p_1p_2p_3p_4$  in four triangles, of which at most one may contain vertices of  $Q$ . If none does, then clearly replacing the intersecting segments with opposite pairs of sides of the quadrilateral  $p_1p_2p_3p_4$  is possible for one of the two choices of opposite sides and will result in a shorter tour.

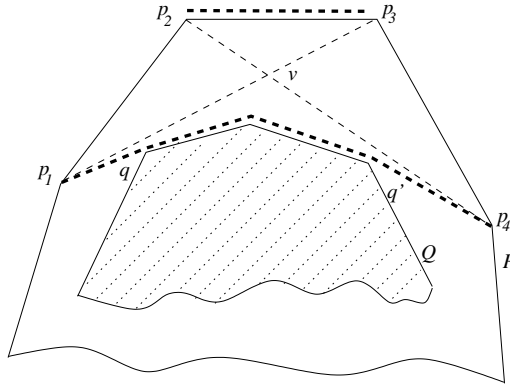


Figure 2: Lemma 2, case 1: all four vertices on  $P$ , using a detour across  $Q$ . The bold dashed line shows one of two shorter tour sections than the original non-bold dashed tour section.

Without loss of generality, then, suppose that the triangle  $p_1vp_4$  contains at least one vertex  $q$  of  $Q$ . Let  $q$  be the first such vertex visible to  $p_1$  and  $q'$  the last visible to  $p_4$ . Then by Proposition 1,  $\|p_1q \oplus q \rightsquigarrow q' \oplus q'p_4\| \ll \|p_1v \oplus vp_4\|$ , so either  $p_1p_2$  and  $p_3p_4$  or else  $p_2p_3$  and  $p_1q \oplus q \rightsquigarrow q' \oplus q'p_4$

results in a shorter tour.

2. **Three vertices are on  $P$  and one on  $Q$ .** Let  $p_1, p_2$  and  $p_3$  denote the three vertices on  $P$  and  $q$  the vertex on  $Q$  such that  $v$  is the point of intersection of  $p_1p_3$  and  $p_2q$ , as for example in Figure 3. As above, the intersecting edges extended to lines divide the plane into four regions, only two of which may contain vertices of  $Q$ . Thus  $p_2$  is visible from both  $p_1$  and  $p_3$ , and  $q$  is visible from at least one of the points  $p_1$  and  $p_3$ .

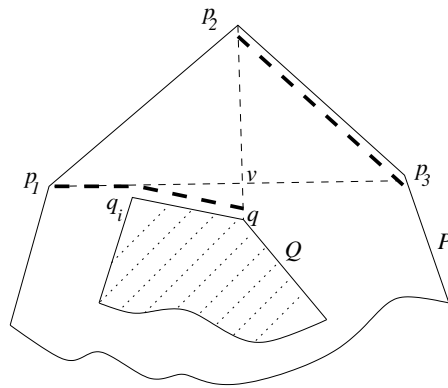


Figure 3: Lemma 2, case 2: three vertices on  $P$ , using a detour across  $Q$ . The bold dashed line shows a shorter tour section than the original non-bold dashed tour section.

If  $q$  is visible from both  $p_1$  and  $p_3$ , we may substitute two opposite edges of the quadrilateral  $p_1p_2p_3q$  as above. If  $q$  is not visible, say from  $p_1$ , then we pick  $q_i$  in the triangle  $p_1vq$  visible from  $p_1$  and note that  $\|p_1q_i\| + \|P_Q(q_i, q)\| < \|p_1v\| + \|vq\|$ . The contradiction follows as above.

3. **Two distinct vertices are on  $P$  and two on  $Q$ .** Let  $p_1, p_2, q_1$ , and  $q_2$  denote the four vertices and suppose  $p_1q_2$  and  $p_2q_1$  intersect at a point  $v$ . Since  $P$  and  $Q$  are convex,  $p_1$  is visible from  $q_1$  and  $p_2$  from  $q_2$ . The contradiction follows as above.
4. **Two vertices are on  $P$  and two on  $Q$ , with  $p_1 = p_2$ .** Consider the point  $p'$  immediately preceding  $p$  on  $P$ . Application of the triangle inequality between  $p'$  and the visible point on  $pq_1$  closest to  $q_1$  (and repeated application as needed using the tangent point of the resulting line segment) yields a shorter path. □

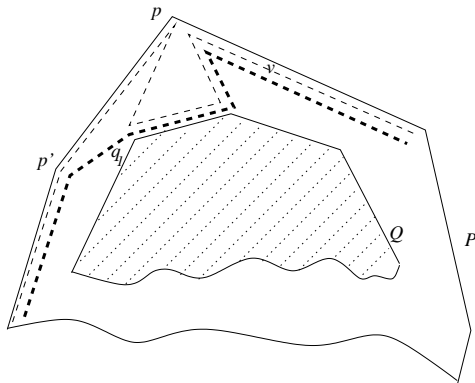


Figure 4: Lemma 2, case 4: one vertex on  $P$ . The bold dashed line shows a shorter tour section than the original non-bold dashed tour section.

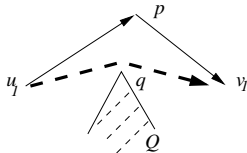


Figure 5: The path  $u_1q_i \rightsquigarrow q_jv_1$  is a shorter path than  $u_1pv_1$ .

**Lemma 3** *There exists a shortest tour  $T$  that visits each  $P$ -vertex exactly once.*

**Proof:** Assume the contrary and let  $p \in P$  be a node visited more than once in  $T$ . Imposing an orientation on the tour  $T$ , consider the last two visits of  $T$  to  $p$ . Specifically, let  $T$  enter  $p$  through  $u_1$  and follow on to  $v_1$  in the first visit, and use pair  $u_2$  and  $v_2$  in its second visit to  $p$ .

Observe that the pairs  $\{u_1, u_2\}$  and  $\{v_1, v_2\}$  are not necessarily distinct and that the pairs  $\{u_1, v_1\}$  and  $\{u_2, v_2\}$  are not visible, otherwise we can avoid one of the last visits while obtaining a shorter tour.

Note further that one vertex from each pair,  $\{u_1, v_1\}$  and  $\{u_2, v_2\}$ , must be on  $P$ , for if, say,  $u_1$  and  $v_1$  were both on  $Q$ , a shorter tour would simply stay on  $Q$  and skip  $p$ . Without loss of generality, then, suppose that  $u_1$  is on  $P$  while  $v_1$  is on either  $P$  or  $Q$  but not visible from  $u_1$ . Let  $q_i$  be the occluding point for  $u_1$  and  $q_j$  for  $v_1$  (where we let  $q_j = v_1$  if  $v_1$  is on  $Q$ ). Then  $u_1q_i \rightsquigarrow q_jv_1$  is a shorter path than  $u_1pv_1$ , cf. Figure 5.  $\square$

**Corollary 4** *There exists a shortest tour  $T$  on which  $P$ -vertices preserve their cyclic order.*

In particular, Corollary 4 says that we may orient a tour  $T$  on  $P$ . Henceforth we will consider  $T$  to be clockwise on  $P$  unless otherwise indicated.

Consider a pair of consecutive vertices  $p_k$  and  $p_k^+$  that are not consecutive on a shortest tour  $T$  satisfying the conditions of the above lemmas. From Lemma 2 and the convexity of  $Q$ , then, we may conclude that  $T$  leaves  $p_k$  through a cross-over edge  $p_k q_i$ , visits some  $Q$ -vertices, and returns to  $P$  via a cross-over edge  $q_j p_k^+$ ; i.e.,  $p_k q_i \rightsquigarrow q_j p_k^+$ , where the path from  $q_i$  to  $q_j$  may be clockwise, counterclockwise, or even retrace some part of its path  $Q$ . Note that  $q_i$  and  $q_j$  need not be distinct.

**Corollary 5** *Other than backtracking, a shortest tour visits vertices of  $Q$  in cyclic order as well.*

**Proof:** Suppose a shortest tour  $T$  containing a clockwise portion  $q_i \rightsquigarrow q_j$  on  $Q$  and a counter-clockwise portion  $q_a q_{a-1} \oplus \dots \oplus q_{b+1} q_b$ . Since the tour has a single cyclic orientation  $T$  on  $P$ , one of the portions of  $T$  must intersect when connecting to the tour on  $P$ . □

Detours in which the orientation on  $Q$  differs from the orientation on  $P$  are special:

**Lemma 6** *If a shortest path is clockwise on  $P$  but counterclockwise on  $Q$ , then  $T$  has precisely one detour from  $P$  to  $Q$ .*

**Proof:** By the Jordan curve theorem [4] and Lemma 4, the detour on  $Q$  either covers all of  $Q$  or else it leaves vertices on  $Q$  which can not be reached by  $T$  without crossing the given detour. Since  $T$  visits all vertices of  $P$  and  $Q$ , the single detour covers all of  $Q$ . □

## 4 Algorithm

By Lemma 6, we must consider two cases: the shortest tour with same orientation on  $P$  and  $Q$  and the shortest tour with opposite orientations. Suppose first that the tours have the same orientation.

Let  $G = (V, E)$  be a directed weighted graph with  $V = \{x_1, \dots, x_m, y_1, \dots, y_m\}$  conceptually equal to the vertices of  $Q$  repeated twice, with  $q_i$  represented at  $x_i$  and again at  $y_i$ . The edges  $(v_i, v_j)$  have weight  $w(v_i, v_j)$ , where  $w(x_i, y_j) = c_{i,j}$  and  $w(y_i, x_j) = 0$  if  $j = i$  or  $j = i + 1$  and  $\infty$  otherwise, where subscripts are interpreted modulo  $m$  and  $c_{i,j}$  is the cost of the cheapest detour  $d_{i,j}$ , as defined in Section 2. Then one of the  $m$  shortest paths from  $x_i$  to  $y_{i-1}$  corresponds to a shortest obstacle avoiding tour of  $P$  and  $Q$ :

**Theorem 7** *For each  $1 \leq h \leq m$  let  $\pi_h$  be the shortest path from  $x_h$  to  $y_{h-1}$ . Let  $\pi$  be the  $\pi_h$  with shortest length. Then  $\pi$  corresponds to a shortest same-orientation obstacle avoiding tour through all  $P$  and  $Q$  vertices.*

**Proof:** By Lemmas 2 and 3, the path  $T$  corresponding to  $\pi$  is a weakly-simple tour and may be thought of as a cyclic tour of  $P$  plus the cost of one or more detours to  $Q$ . The detour  $d_{i,j}$  is represented in the graph  $G$  as edges  $(x_i, y_j)$ . The continuation along  $P$  until the next detour is represented by one of the zero weight edges from  $y_j$  to  $x_j$  or  $x_{j+1}$ . If a tour of shorter length on the vertices of  $P$  and  $Q$  existed, it would have a representation as a shortest path between some  $x_{h'}$  and  $y_{h'-1}$  in  $G$ . By construction, therefore,  $T$  is the shortest same-orientation tour as claimed. It has length  $|P| + |\pi_h|$ .  $\square$

Suppose, on the other hand, that the tour on  $Q$  has opposite orientation than on  $P$ . By Lemma 6, a shortest tour  $T$  will omit precisely one edge of  $P$  and one edge of  $Q$ . We need, therefore, only consider the shortest detour for each candidate edge of  $Q$ . The shortest tour, then, is the shorter of the shortest same-orientation tour and the shortest opposite-orientation tour.

## 5 Complexity

The algorithm in Section 4 leads us down two paths: finding the shortest same-orientation tour and finding the shortest opposite orientation tour. In the former case, we first find the least cost detours for each pair of  $Q$  vertices, then we find  $m$  shortest paths in a graph on  $2m$  vertices. In the latter case we find and compare

$m$  shortest detours.

**Theorem 8** *Finding the shortest tour of  $P$  and  $Q$  vertices requires  $O(m^2 \lg m + m^2 n)$  time and  $O(nm + m^2)$  space.*

**Proof:** Computing the visibility graph of the vertices of  $P$  and  $Q$  takes time and space  $O(mn)$ , since we must consider and store  $mn$  pairs while the supporting tangents can be computed in amortized constant time due to the convexity of  $P$  and  $Q$  using calipers [6].

Since we must consider  $\Theta(nm^2)$  detours, the total cost of computing detours is  $\Theta(nm^2)$ . We use  $O(m^2)$  space to store the values, since we minimize  $c_{i,j}^k$  over  $k$  and so only store the  $c_{i,j}$  and  $d_{i,j}$ .

Finding the shortest same-orientation detours requires  $O(m^2 \lg m)$  time and  $O(m^2)$  space to find the  $m$  shortest paths [3]. (Note that Karger et al.'s  $m^* = O(m)$  in our application.) The total complexity is thus  $O(m^2 \lg m + m^2 n)$  time and  $O(nm + m^2)$  space.

Finding the shortest opposite-orientation tour, from the above, requires  $O(nm + m^2 n)$  time and  $O(mn + m^2)$  space, since we must still compute the visibility graph and the  $d_{i,j}$ 's.

To find the shortest tour, we must find each of the above, and the result follows.  $\square$

To the best of our knowledge this is the first polynomial algorithm for this problem. The more general case of  $k$  polygons (for fixed  $k \geq 2$ ) remains open. No lower bound for the problem is known beyond the trivial linear time needed to look at every vertex.

## 6 Acknowledgements

Jeff Abrahamson and Ali Shokoufandeh gratefully acknowledge the support of the Office of Naval Research grant ONR-N000140410363.

## References

- [1] M. Cutler, Efficient special case algorithms for the  $n$ -line planar traveling salesman problem, *Networks* **10** (1980) 183-195.
- [2] V.G. Deineko, R. van Dal and G. Rote, The convex-hull-and-line traveling salesman problem: a solvable case, *Inf. Proc. Lett.* **51** (1994) 141-148.
- [3] D. R. Karger, D. Koller, and S. J. Phillips. Finding the Hidden Path: Time Bounds for All-Pairs Shortest Paths. *SIAM Journal of Computing*, 22, 1199–1217, December, 1993.
- [4] J. Munkres, *Topology: A First Course*, Prentice-Hall, 1975.
- [5] G. Rote, The  $N$ -line traveling salesman problem, *Networks* **22** (1992) 91-108.
- [6] G. T. Toussaint. Solving geometric problems with the rotating calipers. In *Proceedings of IEEE MELECON 1983*, Athens, Greece, May 1983. (1962), 11-12.