1. Let \( p(x) = x^3 - x + 1 \) and let \( a \) be a root of \( p(x) \).
   (a) Find the multiplicative inverse of \( 1 - 2a + 3a^2 \) in \( \mathbb{Q}(a) \).
   (b) Let \( b = 2 - 3a + 2a^2 \). Find the irreducible polynomial of \( b \) over \( \mathbb{Q} \).

2. Prove that regular polygon of seven sides is not constructible. (Hint: Write \( 2 \cos(\frac{2\pi}{7}) = e^{2\pi i/7} + e^{-2\pi i/7} \) and prove that \( 2 \cos(\frac{2\pi}{7}) \) satisfies \( x^3 + x^2 - 2x - 1 \).

3. If \( \alpha_1, \alpha_2, \alpha_3, \) and \( \alpha_4 \) are the roots of the fourth-degree polynomial \( x^4 + 7x^2 - 6x + 1 \).
   (a) Find the fourth-degree polynomial over \( \mathbb{Q} \) whose roots are \( \alpha_1^2, \alpha_2^2, \alpha_3^2, \) and \( \alpha_4^2 \).
   (b) Find the value of \( \alpha_1^2 \alpha_2^2 + \alpha_1^2 \alpha_3^2 + \alpha_1^2 \alpha_4^2 + \alpha_2^2 \alpha_3^2 + \alpha_2^2 \alpha_4^2 + \alpha_3^2 \alpha_4^2 \).

4. Given the polynomial \( p(x) = x^4 - 5 \). Our goal is to construct and identify the Galois group \( G \) of \( p(x) \) over \( \mathbb{Q} \). Since each group element is a field automorphism, we can think of it as a permutation on the roots of \( p(x) \). For instance, let the roots be denoted by \( \alpha_1, \alpha_2, \alpha_3, \) and \( \alpha_4 \). Then \( \sigma( \alpha_i ) \) must also be a root of \( p(x) \). Let \( \sigma( \alpha_i ) = \alpha_j \) for \( 1 \leq i, j \leq 4 \). Thus \( \sigma \) in considered as the permutation \( \alpha_i \rightarrow \alpha_j \), for \( 1 \leq i, j \leq 4 \). In this way the Galois group \( G \) is identified as a subgroup of \( S_4 \).
   (a) Find the cycle structure for each element of \( G \).
   (b) Determine \( |G| \) and identify the isomorphic structure of \( G \), (i.e. \( G \) is isomorphic to what kind of known group.)

5. Do the same as in problem 4 for the polynomial \( q(x) = (x^2 - 3)(x^2 + 1)(x^3 - 1) \).