1. (Verifying Green's theorem)
Let \( P(x,y) = x^2y, \) \( Q(x,y) = e^x y^2 + 1, \) and the contour \( C \) : Start from the origin \( 0,0 \) going along the curve \( y = x^3 \) to reach the point \( 1,1 \), then return to the origin by moving along the parabola \( y = \sqrt{x} \).
(a) Compute the line integral \( \int_C P \, dx + Q \, dy \) by direct evaluation. The contour \( C \) is decomposed into path \( C_1 \) from the origin \( 0,0 \) to the point \( 1,1 \) along \( y = x^3 \), followed by path \( C_2 \) which is the remaining part of \( C \). You must parametrize \( C_1 \) and \( C_2 \) first. Convert \( \int_{C_1} P \, dx + Q \, dy \) into an ordinary integral and then call Maple for its numerical evaluation. What is the value of \( \int_{C_1} P \, dx + Q \, dy \)?
(b) What is the value of \( \int_{C_2} P \, dx + Q \, dy \)?
(c) The integral \( \int_C P \, dx + Q \, dy \) is just the sum of
\[ \int_{C_1} P \, dx + Q \, dy + \int_{C_2} P \, dx + Q \, dy. \]
What is the value of \( \int_C P \, dx + Q \, dy \)?
(d) Using Green's theorem to obtain the double integral and then convert the double integral into an iterated integral. Then call Maple for its numerical evaluation. Note that you should get the same numerical answer as in part (c).

2. (Verifying Cauchy integral theorem)
(a) Consider the contour integral \( \int_C \frac{e^z}{z^2+9} \, dz \) where the contour \( C \) : Start from 0 going along the curve \( y = x^3 \) to reach the point \( 1+i \), then return to the origin by moving along the parabola \( y = \sqrt{x} \).
To do direct evaluation for the integral \( \int_C \frac{e^z}{z^2+9} \, dz \) we need to break the contour \( C \) into two
parts $C_1$ and $C_2$, where $C_1$ moves from 0 to 1 + $i$ along the curve $y : x^3$, and $C_2$ comes back to 0 along the other parabola. That is,

$$\int_C \frac{e^z}{z^2 + 9} \, dz \equiv \int_{C_1} \frac{e^z}{z^2 + 9} \, dz + \int_{C_2} \frac{e^z}{z^2 + 9} \, dz.$$ 

To evaluate $\int_{C_1} \frac{e^z}{z^2 + 9} \, dz$ we parametrize $C_1 : z(t) : t + it^3, 0 \leq t \leq 1$. This gives $dz : 1 + 3it^3 \, dt$. And insert this parametrization into the integral for evaluation. In this way $\int_{C_1} \frac{e^z}{z^2 + 9} \, dz$ is converted into an ordinary integral with complex integrand

$$\int_0^1 \frac{e^{t + it^3}}{1 + it^3 + 9} \, dt + 3it^3 \, dt.$$ 

We now call for Maple for evaluating $\int_0^1 \frac{e^{t + it^3}}{1 + it^3 + 9} \, dt$. This is how $\int_{C_1} \frac{e^z}{z^2 + 9} \, dz$ is computed directly. What is the value of $\int_{C_1} \frac{e^z}{z^2 + 9} \, dz$?

(b) What is the value of $\int_{C_2} \frac{e^z}{z^2 + 9} \, dz$?

(c) The original contour integral $\int_C \frac{e^z}{z^2 + 9} \, dz$ is the sum of them. What is the value of $\int_C \frac{e^z}{z^2 + 9} \, dz$? Does the result of your calculation for $\int_C \frac{e^z}{z^2 + 9} \, dz$ agree with what Cauchy integral theorem predicts?

3. (Verify Cauchy integral formula)

We now change the contour $C$ in the integral in problem 2 to the circle $|z| : 4$ and still use the same integrand $\frac{e^z}{z^2 + 9}$. That is, we want to compute the contour integral $\int_{|z| : 5} \frac{e^z}{z^2 + 9} \, dz$. An obvious parametrization for the contour is $z(t) : 5e^{it}, 0 \leq t \leq 2\pi, \mathbb{Z}$. We want to use this to carry out a direct numerical evaluation using Maple.

(a) What is the result of your numerical calculation for the integral $\int_{|z| : 5} \frac{e^z}{z^2 + 9} \, dz$?

(b) Using Cauchy Integral Formula to calculate the theoretical value of the integral. What is the theoretical value of the integral? Your numerical computation need to confirm this theoretical value.