Solution for Math 279 Vector and Complex Analysis
Test 1  July 2003

1. Find the directional derivative of the scalar field $f$ at the point $P$ in the direction toward the point $\hat{1}, 0, ?1P$

$$\nabla f_{x,y,z} (\hat{1}, 0, ?1) = x^2 \hat{i} + y \hat{j}$$

Solution: The direction vector is

$$\hat{1} = 1, 0, ?1 \hat{E} = ?1, 3, 0 \hat{E} = ?2, ?3, ?1 \hat{E}$$

so the unit direction vector is

$$\mathbf{b} = \frac{2i + 3j + k}{\sqrt{2^2 + 3^2 + 1^2}} = \frac{2i + 3j + k}{\sqrt{14}}$$

Now

$$D_{\mathbf{b}} \nabla f_{1, 0, ?1} = \hat{f} \nabla f_{1, 3} \hat{i} + 6\hat{b}$$

$$\hat{f} \nabla f_{1, 3} = e^x \hat{i} + e^y \hat{j} \hat{i} + e^z \hat{k}$$

Consequently,

$$D_{\mathbf{b}} \nabla f_{1, 0, ?1} = \hat{1} \hat{f} \hat{i} + j \hat{k} \hat{b} 6 \hat{b} \frac{2i + 3j + k}{\sqrt{14}}$$

$$= \frac{1}{\sqrt{14}} \hat{y} + \frac{3}{\sqrt{14}} \hat{y}$$

2. (30 points) Find the value of the line integral.

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where the vector field $\mathbf{F}_{x,y,z} = x^2 \hat{i} + e^y \hat{j}$, and the curve $C$ is the helix defined by

$$\mathbf{r} \hat{P} = \cos(t) \hat{i} + \sin(t) \hat{j} + 3tk, \quad 0 \leq t \leq 2 \times \frac{Z}{2}.$$

Solution: Note that

$$\nabla f_{x,y,z} = \frac{1}{6} x^2 + e^y$$

Hence by the fundamental theorem of line integral we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \hat{r} \nabla f \cdot d\mathbf{r}$$

$$= \int_C \mathbf{F} \cdot \frac{Z}{2} \hat{P} = \int_C \mathbf{F} \cdot \frac{Z}{2} \hat{P}$$

We now compute $\mathbf{r} \frac{Z}{2} \hat{P}$ and $\frac{Z}{2} \hat{P}$

$$\mathbf{r} \frac{Z}{2} \hat{P} = \cos \frac{Z}{2} \hat{i} + \sin \frac{Z}{2} \hat{j} + \frac{3Z}{2} \hat{k}$$

$$\frac{Z}{2} \hat{P} = \cos 0 \hat{i} + \sin 0 \hat{j} + 0 \hat{k}$$
\[ \dot{X} \cdot \nabla f = 6 \dot{r} : f(0, 1, \frac{3z}{2}) \nabla f(1, 0, 0) \]

\[ = (0 + e^1) \dot{r} \cdot (\dot{y} \frac{1}{6} + 1) \dot{r} \]

\[ = e \cdot \frac{1}{6} \]

3. Let

\[ f(x, y, z) = xe^z \hat{i} + yk \]

Determine the following scalar field:

\[ \text{div} \nabla \text{Curl} f \]

Solution: By vector identity we get

\[ \text{div} \nabla \text{Curl} f = 0 \]

4. (a) Determine if the following line integral is path-independent.

\[ \oint_C xy \, dx + x^2y \, dy \]

where \( C \) is the portion of the parabola \( y = 2x^2 \) from \( \dot{y} = 1, 2 \) to \( \dot{y} = 1, 2 \)

(b) Evaluate the line integral.

Solution:

a) \( \frac{\partial f}{\partial x} \bigg|_{x=0} = \frac{\partial f}{\partial y} \bigg|_{y=0} = 2xy \Leftrightarrow x \not\equiv 0 \), Hence it is not path-independent.

b) To parametrize the curve let

\[ \left\{ \begin{array}{l}
 x : t \\
 y : 2t^2 \\
 \end{array} \right., \quad t \in [0, 1] \]

\[ \text{Plug these in the line integral to get} \]

\[ \oint_C xy \, dx + x^2y \, dy : \oint_{t=1} \dot{y} t^2 \, dt + t^2 \dot{y} t^2 \Phi(t) \, dt = 0 \]