Solution for Math 279 Vector and Complex Analysis
Test 2 July 2003
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1. Use the divergence theorem to evaluate the flux \[ \Phi_F \] of \( F \) through the closed surface \( S \).

\[ F \hat{\mathbf{r}}, y, z \mathcal{P} : \mathbb{R}^2x^2, x, 3y^2 \hat{a} \]

and \( S \) is the four triangular faces of the tetrahedron shown. You must find the value of the integral.

Solution: Since \( \text{div} F = 4x \), by Divergence Theorem we have

\[ \Phi_F \begin{align*} \int_{S} F \cdot n \, dA \quad &\quad \int_{D} \text{div} F \, dv \\ \int_{B} 4xdv \quad &\quad \int_{D} 4xzdA \\ \int_{0}^{y} \int_{\frac{y}{2}}^{1} 12x\hat{y} \quad &\quad \int_{0}^{2} \int_{\frac{y}{2}}^{1} 12x\hat{y} \, dx \\ \int_{0}^{1} \int_{0}^{\frac{y}{2}} 1 \end{align*} \]

, where \( D \) is the triangular region in the \( x \) ? \( y \) plane bounded by the line \( x + \frac{y}{2} = 1 \), the positive \( x \) ?axis and the positive \( y \) ?axis. Carrying out the \( z \) ?integration we have

\[ \int_{0}^{1} \int_{0}^{\frac{y}{2}} 1 \]

or in terms of different order of integration

\[ \int_{0}^{1} \int_{0}^{\frac{y}{2}} 1 \]

2. Use Stokes’s theorem to evaluate the circulation \[ [C F] \] of \( F \) along the path \( C \).

\[ F \hat{\mathbf{r}}, y, z \mathcal{P} : \mathbb{R}^2x^2, x, 3y^2 \hat{a} \]

where \( C \) is the slanted triangular path shown in problem 1 i.e., \( C \) is the triangle oriented from \( \hat{y}, 0, 0 \mathcal{P} \) to \( \hat{y}, 0, 3 \mathcal{P} \) to \( 0, 2, 0 \mathcal{P} \) and back to \( \hat{y}, 0, 0 \mathcal{P} \). Just leave your answer in terms of an iterated integral with limits of integration \emph{explicitly} determined. No evaluation of the integral is required.

Solution:

Compute first

\[ \text{Curl} F : \begin{vmatrix} i & j & k \\ \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \\ 2x^2 & x & 3y^2 \end{vmatrix} \]
By Stokes’ theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \text{curl} \mathbf{F} \cdot d\mathbf{A},$$

where the surface $S$ is chosen as the slanted triangular face as shown. We now parametrize $S$:

$$S : \begin{cases} x : u \\ y : v \\ z : 3 - 2u + v \end{cases}, \quad \mathbf{r} u = 3\mathbf{k}, \mathbf{r} v = \frac{3}{2}\mathbf{j} + 3\mathbf{i}$$

$$\mathbf{r} u \times \mathbf{r} v = \mathbf{k}$$

$$\int_S \text{curl} \mathbf{F} \cdot d\mathbf{A} = \int_D \mathbf{F} \times \mathbf{n} \cdot d\mathbf{A}$$

$$\int_S \text{curl} \mathbf{F} \cdot d\mathbf{A} = \int_D (18v + 0) \, dudv = 13$$

3. Find the moment of inertia of the lamina $S$ of density 1 about the axis $A$.

$S$ : the slanted triangular face shown in problem 1

$A$ : the $z$ axis

Just leave your answer in terms of an iterated integral with limits of integration explicitly determined. No evaluation of the integral is required.

Solution: Since $I_S = \int_S \mathbf{x}^2 + 2y^2 \, d\mathbf{A}$ and the surface $S$ is the same surface as in problem 2. We shall use the same parametrization for $S$ here.

Thus

$$dA : ||\mathbf{r}_u \times \mathbf{r}_v|| \, dudv$$

$$: \left| \left| 3\mathbf{i} + \frac{3}{2}\mathbf{j} + \mathbf{k} \right| \right| \, dudv$$

$$: \sqrt{9 + \frac{9}{4}} + 1 \, dudv$$
\[ \frac{7}{2} dudv \]

or

\[ \int_{s}^{p} \alpha^2 + y^2 dA : \int_{s}^{p} \alpha^2 + v^2 dy \frac{7}{2} dudv \]

\[ \frac{7}{2} \int_{0}^{2} \int_{0}^{2} \hat{u}^2 + v^2 dudv : \frac{35}{12} \]

or

\[ \frac{7}{2} \int_{0}^{2} \int_{0}^{2} \hat{u}^2 + v^2 dvdu : \frac{35}{12} \]