Math 279 Vector and Complex Analysis
Solution for Test 2
February 2004

Dr. William Goh
Name: ______________________________

**In the solutions below the pictures are not drawn. They are referred to the original graphs given in the test.

1. Use Stokes’ theorem to calculate the circulation of the vector field $F$ along the given close curve $C$. You can leave your answer in the form of an iterated integral where the limits of integrations are specified. Evaluation for the integral is not required.

$$F(x, y, z) = x^2i + 3y^2k$$

Solution:

Stoke’s theorem asserts that

$$\int_C F \cdot dr = \int \int_S \text{curl} F \cdot \mathbf{n} ds.$$  

To evaluate the surface integral we need to parametrize $S$:

$$S: \left\{ \begin{array}{ll}
  x = s \\
  y = t \\
  z = 1 - s - t,
\end{array} \right.$$  

where $(s, t)$ is in the triangular region $R$ bounded by the lines: $s + t = 1$, $s = 0$, $t = 0$. Now

$$r = si + tj + (1 - s - t)k$$

$$\Rightarrow r_s = i - k, r_t = j - k$$

$$\Rightarrow r_s \times r_t = (i - k) \times (j - k) = k + j$$

$$\Rightarrow \mathbf{n} dS = (i + j + k) dA$$

$$\text{curl} F = \begin{vmatrix}
  i & j & k \\
  \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
  x^2 & 0 & 3y^2
\end{vmatrix} = i(6y) - j(0) + k(0) = 6yi$$

Hence

$$\int \int_S \text{curl} F \cdot \mathbf{n} ds = \int \int_R (6yi) \cdot (i + j + k) dA = \int \int_R 6ydA$$

$$= 6 \int_0^1 \int_0^{1-s} t dt ds = 1$$

2. Use Divergence theorem to calculate the flux of the vector field $F$ through the given surface $S$. 

Evaluation for the integral is required.

\[ \mathbf{F}(x, y, z) = 4x \mathbf{i} + x^2y \mathbf{j} + x^2z \mathbf{k} \]

and the surface \( S \) is the surface of the tetrahedron shown.

**Solution:**

By the divergence theorem

\[
\int_S \mathbf{F} \cdot d\mathbf{s} = \int_B \text{div} \mathbf{F} \, dv
\]

\[
= \int_B \int (4 + 2x^2) \, dv = \int_R \int_0^{2(1-x-y)} (4 + 2x^2) \, dz \, dA
\]

\[
= \int_R 2(4 + 2x^2)(1 - x - y) \, dA,
\]

where \( R \) is the region bounded by \( x + y = 1, x = 0, y = 0 \). The above double integral is converted into

\[
= 2 \int_0^1 \int_0^{1-x} (4 + 2x^2)(1 - x - y) \, dy \, dx
\]

\[
= 2 \int_0^1 (4 + 2x^2)((1 - x)y - \frac{y^2}{2})|_0^{1-x} \, dx
\]

\[
= 2 \int_0^1 (4 + 2x^2)(\frac{(1 - x)^2}{2}) \, dx
\]

\[
= \int_0^1 (4 + 2x^2)(1 - x)^2 \, dx
\]

\[
= \int_0^1 (2x^4 - 4x^3 + 6x^2 - 8x + 4) \, dx
\]

\[
= (\frac{2}{5}x^5 - x^4 + 2x^3 - 4x^2 + 4x)|_0^1
\]

\[
= \frac{7}{5}
\]

3.

Use Green’s theorem to evaluate the integral \( \oint_C \mathbf{F} \cdot d\mathbf{r} \).

\[ \mathbf{F}(x, y) = \tan(0.2x) \mathbf{i} + x^3 \mathbf{j} \]

and curve \( C \) is given as shown. Evaluation for the integral is required.

**Solution:**

\[
\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_C \tan(0.2x) \, dx + (x^3) \, dy
\]

\[
= \int \int_R 5x^4 \, dA,
\]
where $R$ is the semicircular disc shown in the test.

We now use the polar coordinates to evaluate the integral. Thus

$$
\int_0^\pi \int_0^5 5(r \cos \theta)^4 (r \sin \theta) (r dr d\theta)
$$

$$
= 5 \int_0^\pi \int_0^5 r^6 \cos^4 \theta \sin \theta dr d\theta
$$

$$
= \frac{5^8}{7} \int_0^\pi \cos^4 \theta \sin \theta d\theta.
$$

Use the substitution

$$
u = \cos \theta
$$

$$
du = -\sin \theta d\theta
$$

$$
\frac{5^8}{7} \int_0^\pi \cos^4 \theta \sin \theta d\theta = \frac{5^8}{7} \int_{-1}^1 u^4 du = \frac{5^8}{7} \cdot \frac{2}{5}
$$

$$
= \frac{2 \cdot 5^7}{7} = \frac{156250}{7} \approx 22321.
$$

4.

Calculate the surface integral

$$
\int \int_S zye^{-xz} dS,
$$

where $S$ is given as shown. You may leave your answer in the form of an iterated integral where the limits of integrations are specified. Evaluation for the integral is not required.

**Solution:**

We begin with a parametrization of $S$:

$$
S : \begin{cases} 
  x = s \\
  y = t \\
  z = 2(1 - s - t)
\end{cases}
$$

where the $(s, t)$ is in the triangular region $R$ bounded by the lines: $s + t = 1, s = 0, t = 0$. Now

$$
r = si + tj + 2(1 - s - t)k
$$

$$
\mathbf{r}_s \times \mathbf{r}_t = (i - 2k) \times (j - 2k)
$$

$$
= 2i + 2j + k
$$

So the area element $dS$ is

$$
dS = ||\mathbf{r}_s \times \mathbf{r}_t|| dA = 3 dA
$$

Now we insert these in the surface integral to get

$$
\int \int_S zye^{-xz} ds
$$

$$
= \int \int_R 2(1 - s - t)te^{-2s(1-s-t)}(3 dA)
$$
\[ = 6 \int \int_R (1 - s - t)te^{-2s(1-s-t)}dA \]
\[ = 6 \int_0^1 \int_0^{1-s} t(1 - s - t)e^{-2s(1-s-t)}dt ds. \]