• This list of problems are created to represent the spectrum of the topics that we have covered in the quarter. Also it will help you to focus your preparation for the final exam.

• Final exam problems are not identical to this list. They are mild variants.

• Those who study this list carefully and honestly will gain advantages in the final exam and those who don’t will loose advantages in the test.

• We will discuss some of these problems in the recitation of week 10.

• No solution for this mock final will be posted in the home pages.

1 Gradient, Directional derivative

(1) Find the point/points on the given surface at which the tangent plane is parallel to the indicated plane.

\[ z^2 = x^4 + y^2; x + 2y - 3z = 4 \]

(2) Find the directional derivative of \( f(x, y) = x^3 + 2xy \) at the point \((1, 2)\) in the direction of a tangent vector to the graph of \( x^2 + 2y = 6 \) at \((2, 1)\).

2 Curl, Divergence

(1) Calculate the curl and the divergence of the vector field

\[ \mathbf{F}(x, y, z) = 2\sin xy \mathbf{i} + e^{yz} \mathbf{j} + 3y^2 \mathbf{k} \]

at the point \((1, 2, -1)\).

(2) Prove the following two identities:

(a) \( \text{curl}(\text{grad } f) = \mathbf{0} \)

(b) \( \nabla \cdot (\nabla \times \mathbf{F}) = 0 \)

3 Definition of line integral, Independence of path

(1) Evaluate the line integral \( \int_C 3ydx + 2xdy \), where \( C \) is given by \( x = y^3 + 1 \) from \((0, -1)\) to \((9, 2)\).

(2) Evaluate the line integral \( \int_C ydx + zdy + xdz \), where \( C \) is given by \( x = 3t, y = t^3, z = \frac{3}{2}t^2 \) from \((0, 0, 0)\) to \((6, 8, 5)\).

(3) Show that the given integral is independent of path and evaluate it using a potential function:

\[ \int_{(2,3,1)}^{(1,2,1)} 2xzdx + 2yzydy + (x^2 + y^2)dz \]

(4) Determine if the line integral

\[ \int_C [(x^3 + 2e^{-y})dx + (4y - 2xe^{-y})dy] \]

is independent of path or not, where the contour \( C \) is the curve \( y = x^6 \) with the initial point \((0, 0)\) and end point \((1, 1)\).

(5) Evaluate the above line integral in (a).(Hint: determine a potential function)
4 Double integral, interchange of order of integration, Decomposition of region of integration

(1) Evaluate the given iterated integral:
\[ \int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3 + 1} \, dx \, dy. \]
You may have to interchange the order of integration.

(2) Evaluate the given double integral: \( \int_R x \, dA \), where the region \( R \) is defined by the following inequalities: \( x + y \geq 1, x + y \leq 4, x \geq 0, y \geq 0 \). Here we demand that the \( x \) integration must be performed first.

5 Polar integration

(1) Use polar coordinates to evaluate
\[ \int_0^2 \int_{\sqrt{8-x^2}}^{x+1} \frac{1}{1+x^2+y^2} \, dy \, dx. \]

(2) Use polar coordinates to evaluate
\[ \int_0^\infty \int_0^\infty \frac{1}{(1+x^2+y^2)^2} \, dx \, dy. \]

6 Green’s theorem

(1) Use Green’s theorem to evaluate
\[ \oint_C (e^{x^2} - 2y^3) \, dx + (2x^3 - \sin y) \, dy, \]
where \( C \) is the circle \( x^2 + y^2 = 1 \).

(2) Use Green’s theorem to evaluate
\[ \oint_C (-16y + \sin x^2) \, dx + (4e^y + 3x^2) \, dy, \]
where the trajectory \( C \) is the perimeter of the triangle with \((1, 1), (0, 1)\) and \((1, 0)\) as vertices and is oriented counterclockwise.

7 Surface integral

(1) This problem deals with the conversion of integrals. Let \( C \) be the intersection curve of the surfaces
\[ x^2 + y^2 = 4, z = x^2y. \]

(a) Find a parametrization of the curve \( C \).

(b) Express the line integral \( L = \oint_C z^2 \, dx + xy^2 \, dy + y \, dz \) as an ordinary integral where contour \( C \) is the contour in (a). (No evaluation of the integral is required)

(c) According to Stoke’s theorem \( L \) is equal to a surface integral \( I \). Let \( S \) be the surface in \( I \). Find a parametrization for \( S \).

(d) Express \( I \) as an iterated integral. (No evaluation of the iterated integral is required.)

(2) Compute the surface integral \( \iint_S x^2 \, z \, dS \), where \( S \) is the northern hemispherical surface with radius 1. (Hint: parametrizing \( S \) first)
8 **Stoke’s theorem**

(1) Use Stokes’ theorem to evaluate the circulation

\[
\oint_C (2z + x)dx + (x^2 - z)dy + (x^2 + y)dz,
\]

where \( C \) is the triangle with vertices \((1, 0, 0), (0, 1, 0), (0, 0, 1)\).

(2) Find the circulation of the vector field

\[
\mathbf{F} = z^2e^{x^2}i + xy^2j + \tan^{-1}y k
\]

along the curve \( C \), where \( C \) is the intersection of the plane \( z + y = 5 \) with the cylinder \( x^2 + y^2 = 1 \).

9 **Gauss Divergence theorem**

(1) Find the flux \( \iint_S \mathbf{F} \cdot \mathbf{n} \, dA \) of the vector field

\[
\mathbf{F} = y^3i + x^4j + z^3k
\]

out of the following closed surfaces:

(a) closed unit spherical surface: \( x^2 + y^2 + z^2 = 1 \).

(b) Six faces of the unit cube:

\[
0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1.
\]

(2) Find the flux \( \iint_S \mathbf{u} \cdot \mathbf{n} \, dA \) of the vector field \( \mathbf{u} = xi + j \) out of the closed surfaces \( S \), where \( S \) is the surface (including two lids) of the cylinder

\[
0 \leq x^2 + y^2 \leq 1, 0 \leq z \leq 2
\]

10 **Polar form, Euler identity**

(1) Calculate \((1 + i)^{122}\).

(2) Calculate \((1 - i)^{\frac{1}{3}}\)

11 **Analyticity, Cauchy-Riemann equations**

Determine whether each of the following two functions is analytic or not. In case of analytic function also find its derivative and express the result in terms of variable \( z \). Here \( z = x + iy \)

(a) \( f(z) = \Re z + \Re \)

(b) \( g(z) = e^{-i(x-iy)} + e^{-i(x+iy)}\)

(c) \( h(z) = \frac{x}{x^2+y^2} + i \frac{y}{x^2+y^2}\)

12 **Complex logarithm, Sine, Cosine, Exponential, and General Power Function**

(1) Find all values of \((1 - i)^i\)

(2) Find \( \Re(\sin(3 + 2i))\)

(2) Find all values of \( z \) satisfying the equation \( \cos z = 10 \).

(3) Find all values of \( z \) satisfying the equation \( e^{2z - 1} = 2i \).
13 Complex contour integral

Use the definition of complex contour integral to evaluate the following contour integrals:

(a) $\int_{|z|=2} \frac{1}{z} \, dz$ (Hint: parametrizing the contour first)

(b) $\int_C \text{Re}(z) \, dz$, where $C$ is the parabolic curve running from $(0, 0)$ to $(1, 1)$ along $y = x^2$.

(c) $\int_C \frac{1}{z} \, dz$, where $C$ is the line segment from $1 + i$ to $2 - i$.

(d) $\int_C z e^{-z} \, dz$, where $C$ is the line segment from $1 + i$ to $2 - i$.

(e) $\int_C (\frac{1}{z} + z) \, dz$, where $C$ is the line segment from $1 + i$ to $2 - i$.

14 Cauchy integral formula, Residue theory

(1) Evaluate the integral $\int_C \frac{e^z}{z(2-z)} \, dz$ and express the answer in term of its real part and imaginary part. Here the contour $C$ is defined as $|z| = 10$.

(2) Evaluate the integral $\int_C \frac{3z}{(e^z - 2)^2} \, dz$, where $C$ is the contour $|z| = 2$.

(3) Evaluate the integral $\int_C \frac{z}{z^2 - 1} \, dz$, where $C$ is the contour $|z| = 2$.

(4) Evaluate the integral $\int_C \frac{3z^2 + 2}{z^2} \, dz$, where $C$ is the contour $|z| = \frac{1}{2}$.

(5) Evaluate the integral $\int_{-\infty}^{\infty} \frac{1}{1 + x^4} \, dx$.

(6) Evaluate the integral $\int_0^{2\pi} \frac{1}{3 + \sin z} \, dx$.