Eigenvalues and Eigenvectors

Look at $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

Check it out:

$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Both calculations show the pattern

$Ax = \lambda x$

for a scalar $\lambda$ and nonzero vector $x$.

- $\lambda$ is called an eigenvalue of $A$
- $x$ is called an associated eigenvector of $\lambda$
Problem: Given a matrix $A$, how does one go about getting the eigenvalues and eigenvectors?

The clue

$$Ax = \lambda x$$

or

$$Ax = \lambda Ix$$

or

$$(A - \lambda I)x = 0$$

An eigenvector must be a nonzero vector, and this means that $(A - \lambda I)$ is NOT invertible, or equivalently,

$$\det(A - \lambda I) = 0$$

$\text{ <-- one way to get eigenvalues}$

Eigenvectors come later when $(A - \lambda I)x = 0$ is solved.

It is a two-step process.
Example

\[ A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \text{ and } A - \lambda I = \begin{bmatrix} 2 - \lambda & 3 \\ 3 & 2 - \lambda \end{bmatrix} \]

Step 1: Set determinant to zero and solve for roots

\[ |A - \lambda I| = (2 - \lambda)^2 - 9 = \]

simplify and solve

\[ \lambda^2 - 4\lambda + 4 - 9 = \lambda^2 - 4\lambda - 5 = 0 \]

The roots of the polynomial are \( \lambda = 5 \) and \( \lambda = -1 \)

Step 2. Get the two corresponding eigenvectors.
For $\lambda = 5$, find a nontrivial solution of 

$$(A - 5I)p_1 = 0$$

But 

$$A - 5I = \begin{bmatrix} 2 & -5 & 3 \\ 3 & 2 & -5 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$$

and you can reduce the augmented matrix as 

$$\begin{bmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This tells us that the first eigenvector $p_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ satisfies the constraint $a = b$. For convenience, choose $a = 1$ and 

$$p_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ <-- eigenvector corresponding to } \lambda = 5$$
Any multiple of $\mathbf{p}_1$ is also an eigenvector. One standard choice is to rescale the eigenvector so that it is a unit vector.

But $\|\mathbf{p}_1\| = \sqrt{1 + 1} = \sqrt{2}$

and consequently another eigenvector is

$$\mathbf{p}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} .7071 \\ .7071 \end{bmatrix}$$

This is the form you will see when using Matlab or a TI-85.

The other eigenvector associated with $\lambda = -1$ is

$$\mathbf{p}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

or in Matlab form

$$\mathbf{p}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -.7071 \\ .7071 \end{bmatrix}$$