Eigenvalues and Eigenvectors

\[ A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \]

A is symmetric and not invertible because \( \det(A) = 0 \)

Look at the equation \( Ax = \lambda x \) for a nonzero \( x \)

Equivalent to

\[ (A - \lambda I)x = 0 \]

Definition. The number \( \lambda \) is an eigenvalue of \( A \) if and only if

\[ \det(A - \lambda I) = 0 \]

\[ (A - \lambda I) = \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix} \]

\[ \det(A - \lambda I) = (\lambda - 1)(\lambda - 4) - 4 \]

\[ = \lambda^2 - 5\lambda \]

Set to zero to get roots of 0 and 5. Then go for the corresponding eigenvectors.
»A = [1 2;2 4]
A =
  1   2
  2   4

»evals = eig(A); % get eigenvalues
»disp(evals)
  0
  5

»p1 = null(A,'r') % eigenvector for 0
p1 =

   -2
    1

»p2 = null(A - 5*eye(2),'r') % eigenvector for 5
p2 =

   0.5000
   1.0000

»help null  % Simple version

NULL   Null space.
   Z = NULL(A,'r') is a "rational" basis for the null space obtained
   from the reduced row echelon form.
Classical steps for getting eigenvalues/eigenvectors

1. Compute the determinant of $A - \lambda I$. It is a polynomial in $\lambda$
2. Find the roots of this polynomial by solving

$$\det(A - \lambda I) = 0$$

3. For each eigenvalue $\lambda$, solve $(A - \lambda I)x = 0$ to find an eigenvector $x$. There is a different $x$ for each different eigenvalue.

Find the eigenvalues and eigenvectors of $A$ and $A^2$ and $A^{-1}$ and $A + 4I$

when $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$

Eigenvalues of $A$ are 1 and 3 - direct calculation or use `poly` command in MATLAB

```matlab
» poly(A) % MATLAB characteristic polynomial
ans =
   1.0000  -4.0000    3.0000

» poly(A^2)
ans =
   1.0000  -10.0000    9.0000
```
Eigenvector for 1 is \([1 \ 1]^T\)

Eigenvector for 3 is \([1 \ -1]^T\)

Using MATLAB,

```matlab
>> A
A =
    2  -1
   -1   2

>> p1 = null(A - 1*eye(2),'r')
p1 =

    1
    1

>> p2 = null(A - 3*eye(2),'r')
p2 =

   -1
    1
```

There is yet another way to get eigenvalues and eigenvectors using the `eig` command with two outputs.
»[P,D] = eig(A)  % two outputs P and D
P =

-0.7071   -0.7071
0.7071    -0.7071

Normalized eigenvalues are found in the columns of P. Each column is a unit vector but still an eigenvector.

D =

3.0000    0
0        1.0000

Eigenvalues of A are found on the diagonal on D
Show Time

This demo is fun to watch!!

\[
A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}
\]

>> eigshow  \% comes with MATLAB
A GUI appears and what follows are snapshots of the figure window.

Use mouse to move green x vector until it lines up with A*x.
Make $A\mathbf{x}$ parallel to $\mathbf{x}$
Make $A\mathbf{x}$ parallel to $\mathbf{x}$