(10 pts) 1. Find the eigenvalues of $2A$ when $A = \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

The eigenvalues of $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ are 2 and 4. But $2A = \frac{1}{2} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and our work from last week suggests that the eigenvalues of $2A$ are 1 and 2. Just halve the number 2 and 4.

2. The matrix $A = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 16 & 0 & 0 & 16 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The row reduced form of $A$ is $R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and all answers are based on it.

(10 pts) (a) Find a basis for the column space of $A$ and explain your choice.

Pivots occur in columns 1, 2, and 3 of $R$. Thus the first three columns of $A$ are a basis for the column space of $A$.

(10 pts) (b) Construct a basis for the row space of $A$ and explain.

The first three rows of $R$ are a basis for the row space of $A$. Discussed in book.

(10 pts) (c) Find a basis for $\text{Nul } A$.

There are three pivots and one free variable. $\text{Nul } A$ has dimension 1 and a basis can be constructed from $R$ in the usual way. A basis for $\text{Nul } A$ is

\[
\begin{bmatrix} -1 \\ -3 \\ 3 \\ 1 \end{bmatrix}
\]
3. This question deals with an unspecified matrix $A$ with $m$ rows and $n$ columns.

(10 pts.) (a) State the Rank Theorem for $A$. Recall that the Rank Theorem deals with $\text{col } A$, $\text{Nul } A$, and $n$.

The result is $\dim(\text{Col } A) + \dim(\text{Nul } A) = n$ <-- the number of columns in $A$

Mark each statement below as True or False and give reasons for full credit.

(10 pts.) (b) $\dim \text{row } A + \dim \text{Nul } A = n$

This result is true because the row space has the same dimension as the column space as discussed in the book and in class. The statement in nothing more than a rewrite of the theorem on rank.

(10 pts.) (c) $\dim \text{col } A + \dim \text{Nul } A^T = m$

This result is also true and it amounts to a rewrite of the rank theorem for the transpose of $A$. The number of columns in the transpose is $m$ and the dimension of $\text{col } A$ equals the dimension of the row space of $A$ which equals the dimension of the column space of its transpose. In effect, $\dim \text{col } A = \dim \text{col } A^T$.

(10 pts.) 4. Find the eigenvalues of $\begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix}$.

The characteristic polynomial is $\det( A - \lambda I) = \lambda^2 + 4\lambda + 13$ and the quadratic formula or its equivalent gives the roots as $\lambda = -2 \pm 3i$. Note that the roots occur in complex conjugate pairs.

5. The matrix $A= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is known to be similar to the diagonal matrix $D= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(10 pts) (a) What are the eigenvalues of $A$? Explain.

$A$ is similar to $D$, and work in the course indicates immediately that the diagonal elements of $D$ are the eigenvalues of $A$: $-1, 1, 1$ with $1$ being a repeated root.

(10 pts) (b) Find an invertible matrix $P$ that satisfies $AP = PD$.

The second column of $P$ is an eigenvector corresponding to the eigenvalue $-1$ because of $D.$
A + I = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and its row reduced form is } \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \text{ There are two pivots and one free variable. An eigenvector is } \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and this can be used for the second column of } P. \text{ The first and third columns of } P \text{ are a consequence of the eigenvalue of 1 and}

A - I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \text{ Its row reduced form is } \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ - one pivot and two free variables. This means that the null space has dimension two. A standard approach that goes back to chapter one tells us that a basis for the null space is found in the columns of } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \text{ To conclude, the matrix } P \text{ looks like } \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \text{ You could swap columns one and three, but the middle column must be an eigenvector for } -1.