Fourier Approximations for Square Wave

The Fourier approximation using \( n \) cosines and sines is

\[
f_n(t) = \frac{1}{2}a_0 + \sum_{k=1}^{n} \left( a_k \cos(k\omega t) + b_k \sin(k\omega t) \right)
\]

with Fourier coefficients

\[
a_k = \frac{2}{T} \int_{0}^{T} f(t) \cos(k\omega t) dt
\]

\[
b_k = \frac{2}{T} \int_{0}^{T} f(t) \sin(k\omega t) dt
\]

Get Fourier approximations for the square wave of period \( 2\pi \).

![Figure 1](image-url)

Only a part of the wave form is shown. It repeats every \( 2\pi \) units.
\[ \omega = \frac{2\pi}{T} = 1 \]

\[ a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(kt) \, dt \]

\[ b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(kt) \, dt \]

Goal: **Calculate** and **plot** the first few Fourier approximations for the function above.

Set up and calculate the Fourier coefficients

\[ a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(kt) \, dt = 0 \]

\[ b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(kt) \, dt = \frac{2}{\pi} \int_0^{\pi} \sin(kt) \, dt \]

\[ b_k = \frac{-2}{k\pi} \cos(kt) \bigg|_0^\pi = \frac{2}{k\pi} \left(1 - (-1)^k\right) \]

**Even harmonics are zero:**

\[ b_k = \begin{cases} 
\frac{4}{k\pi}, & k \text{ odd} \\
0, & k \text{ even}
\end{cases} \]

\[ b_1 = \frac{4}{\pi} \quad b_3 = \frac{4}{3\pi} \quad b_5 = \frac{4}{5\pi} \]
\[ f_1(t) = \frac{4}{\pi} \sin t \]

Include two harmonics: \[ f_3(t) = f_1(t) + \frac{4}{3\pi} \sin 3t \]
Three harmonics with the others superimposed

$$f_5(t) = f_3(t) + \frac{4}{5\pi} \sin 5t$$
Even better: