Important Relationships

$x$ and $y$ are vectors in $\mathbb{R}^n$

Dot or inner product

$$x \cdot y = x^T y = y^T x = y \cdot x$$

Suppose that the columns of $A$ and $B$ are vectors in $\mathbb{R}^n$.

$$A = [a_1, a_2, a_3] \text{ and } B = [b_1, b_2, b_3].$$

Use partitioned matrix ideas from 2.4,

$$B^T = \begin{bmatrix} b_1^T \\ b_2^T \\ b_3^T \end{bmatrix} \quad \text{ <-- row vectors}$$

What are the typical elements of $B^T A$?

Use block matrix multiplication,

$$B^T A = \begin{bmatrix} b_1^T \\ b_2^T \\ b_3^T \end{bmatrix} [a_1, a_2, a_3] = \begin{bmatrix} b_1^T a_1 & b_1^T a_2 & b_1^T a_3 \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

Each entry is a inner product in disguise.
This is the inner product form of the two-fingered product rule introduced in 2.1.

Run a small experiment to confirm the ideas.

\[ A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = B \]

\[ B' A = \begin{bmatrix} b_1 \cdot a_1 & b_1 \cdot a_2 & b_1 \cdot a_3 \\ b_2 \cdot a_1 & b_2 \cdot a_2 & b_2 \cdot a_3 \\ b_3 \cdot a_1 & b_3 \cdot a_2 & b_3 \cdot a_3 \end{bmatrix} \]

\[ \begin{array}{c}
\text{ans} = \\
5 & 8 & \text{ <-- entries are inner products} \\
0 & 1
\end{array} \]

\[ \text{disp}(\text{dot}(B(:,1),A(:,1))) \]
5
\[ \text{disp}(\text{dot}(B(:,2),A(:,1))) \]
0
\[ \text{dot}(B(:,1),A(:,2)) \]
8
\[ \text{dot}(B(:,2),A(:,2)) \]
1

It checks!!
If $A$ has $m$ rows and $n$ columns and $B$ has $m$ columns,

$$B^T A = \begin{bmatrix}
    b_1 \cdot a_1 & b_1 \cdot a_2 & \cdots & b_1 \cdot a_n \\
    b_2 \cdot a_1 & b_2 \cdot a_2 & \cdots & b_2 \cdot a_n \\
    \vdots & \vdots & \ddots & \vdots \\
    b_m \cdot a_1 & b_m \cdot a_2 & \cdots & b_m \cdot a_n
\end{bmatrix}$$
What about the product $\mathbf{A}\mathbf{B}^T$?

A and B must have the same number of columns.

$$
\mathbf{A}\mathbf{B}^T = 
\begin{bmatrix}
    b_1^T \\
    b_2^T \\
    b_3^T
\end{bmatrix}
\begin{bmatrix}
    a_1, a_2, a_3
\end{bmatrix}
$$

Outer products come to mind

$$
\mathbf{A}\mathbf{B}^T = a_1 b_1^T + a_2 b_2^T + a_3 b_3^T
$$

A matrix product can be described in many different ways.

Run through a small experiment to confirm this result.

»help pascal

PASCAL Pascal matrix.  
PASCAL(N) is the Pascal matrix of order N: a symmetric positive definite matrix with integer entries, made up from Pascal's triangle. Its inverse has integer entries.
```matlab
» A = pascal(3)
A =
1 1 1
1 2 3
1 3 6

» B = inv(A)'
B =
3 -3 1
-3 5 -2
1 -2 1

» ab1 = A(:,1)*B(1,:)
ab1 =
3 -3 1
3 -3 1
3 -3 1

» ab2 = A(:,2)*B(2,:)
ab2 =
-3 5 -2
-6 10 -4
-9 15 -6

» ab3 = A(:,3)*B(3,:)
ab3 =
1 -2 1
3 -6 3
6 -12 6

Add them up and see you get the identity.
```
```matlab
»s2 = ab1+ab2
s2 =
    0  2  -1
   -3  7  -3
   -6 12  -5

»s3 = s2+ab3
s3 =
    1  0  0
    0  1  0
    0  0  1

which agrees with

»A*B
ans =
    1  0  0
    0  1  0
    0  0  1
```