SVD and Image Compression

Look at the SVD of a matrix in outer expansion form

\[ A = \sum_{k=1}^{n} \sigma_k u_k v_k^T \]

The \( \sigma \)'s are arranged in decreasing order.

The matrix to be analyzed:

```matlab
» disp(C)
0  2  -2  0  0  2  -2  0
 2  3  0  -2  2  3  0  -2
-2  0  -3  2  -2  0  -3  2
0  -2  2  0  0  -2  2  0
0  -2  2  0  0  4  -4  0
-2  -3  0  2  4  6  0  -4
2  0  3  -2  -4  0  -6  4
0  2  -2  0  0  -4  4  0
0  2  -2  0  1  2  0  0
2  3  0  -2  2  4  0  0
-2  0  -3  2  0  0  1  2
0  -2  2  0  0  0  2  4

» size(C)
ans =

12  8
```
A list of singular values:

12.2474
11.6342
8.6603
8.0402
5.0000
0.0000
0.0000
0.0000

How well do the partial sums of outer products approximate $C$?

A plot of $d_k = \| C - C_k \|_F$ vs. $k$ follows.
This surface agrees with mesh(C) for all practical purposes.
Look at numbers (harder to interpret)

```matlab
disp(num2str(C1, fmtstr))
```

<p>| | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.80</td>
<td>1.60</td>
<td>-0.40</td>
<td>-0.80</td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>1.60</td>
<td>3.20</td>
<td>-0.80</td>
<td>-1.60</td>
<td></td>
</tr>
<tr>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.40</td>
<td>-0.80</td>
<td>0.20</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.80</td>
<td>-1.60</td>
<td>0.40</td>
<td>0.80</td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>1.60</td>
<td>3.20</td>
<td>-0.80</td>
<td>-1.60</td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>3.20</td>
<td>6.40</td>
<td>-1.60</td>
<td>-3.20</td>
<td></td>
</tr>
<tr>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.80</td>
<td>-1.60</td>
<td>0.40</td>
<td>0.80</td>
<td></td>
</tr>
<tr>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-1.60</td>
<td>-3.20</td>
<td>0.80</td>
<td>1.60</td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.80</td>
<td>1.60</td>
<td>-0.40</td>
<td>-0.80</td>
<td></td>
</tr>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>1.60</td>
<td>3.20</td>
<td>-0.80</td>
<td>-1.60</td>
<td></td>
</tr>
<tr>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.40</td>
<td>-0.80</td>
<td>0.20</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.00</td>
<td>-0.80</td>
<td>-1.60</td>
<td>0.40</td>
<td>0.80</td>
<td></td>
</tr>
</tbody>
</table>

```matlab
disp(num2str(C2, fmtstr))
```

<p>| | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.17</td>
<td>-0.08</td>
<td>0.33</td>
<td>-0.17</td>
<td>0.40</td>
<td>1.80</td>
<td>-1.20</td>
<td>-0.40</td>
<td></td>
</tr>
<tr>
<td>-0.08</td>
<td>0.04</td>
<td>-0.17</td>
<td>0.08</td>
<td>1.80</td>
<td>3.10</td>
<td>-0.40</td>
<td>-1.80</td>
<td></td>
</tr>
<tr>
<td>0.33</td>
<td>-0.17</td>
<td>0.66</td>
<td>-0.33</td>
<td>-1.20</td>
<td>-0.40</td>
<td>-1.40</td>
<td>1.20</td>
<td></td>
</tr>
<tr>
<td>-0.17</td>
<td>0.08</td>
<td>-0.33</td>
<td>0.17</td>
<td>-0.40</td>
<td>-1.80</td>
<td>1.20</td>
<td>0.40</td>
<td></td>
</tr>
<tr>
<td>0.68</td>
<td>-0.34</td>
<td>1.37</td>
<td>-0.68</td>
<td>-0.05</td>
<td>4.02</td>
<td>-4.10</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>-0.34</td>
<td>0.17</td>
<td>-0.68</td>
<td>0.34</td>
<td>4.02</td>
<td>5.99</td>
<td>0.05</td>
<td>-4.02</td>
<td></td>
</tr>
<tr>
<td>1.37</td>
<td>-0.68</td>
<td>2.73</td>
<td>-1.37</td>
<td>-4.10</td>
<td>0.05</td>
<td>-6.19</td>
<td>4.10</td>
<td></td>
</tr>
<tr>
<td>-0.68</td>
<td>0.34</td>
<td>-1.37</td>
<td>0.68</td>
<td>0.05</td>
<td>-4.02</td>
<td>4.10</td>
<td>-0.05</td>
<td></td>
</tr>
<tr>
<td>-0.12</td>
<td>0.06</td>
<td>-0.23</td>
<td>0.12</td>
<td>1.08</td>
<td>1.46</td>
<td>0.17</td>
<td>-1.08</td>
<td></td>
</tr>
<tr>
<td>0.06</td>
<td>-0.03</td>
<td>0.12</td>
<td>-0.06</td>
<td>1.46</td>
<td>3.27</td>
<td>-1.08</td>
<td>-1.46</td>
<td></td>
</tr>
<tr>
<td>-0.23</td>
<td>0.12</td>
<td>-0.47</td>
<td>0.23</td>
<td>0.17</td>
<td>-1.08</td>
<td>1.33</td>
<td>-0.17</td>
<td></td>
</tr>
<tr>
<td>0.12</td>
<td>-0.06</td>
<td>0.23</td>
<td>-0.12</td>
<td>-1.08</td>
<td>-1.46</td>
<td>-0.17</td>
<td>1.08</td>
<td></td>
</tr>
</tbody>
</table>

6 7.4
```matlab
%disp(num2str(C3,fmtstr))
0.97  1.52 -0.07 -0.97  0.40  1.80 -1.20 -0.40
1.52  3.24 -0.97 -1.52  1.80  3.10 -0.40 -1.80
-0.07 -0.97  0.86  0.07 -1.20 -0.40 -1.40  1.20
-0.97 -1.52  0.07  0.97 -0.40 -1.80  1.20  0.40
-0.12 -1.94  1.77  0.12 -0.05  4.02 -4.10  0.05
-1.94 -3.03  0.12  1.94  4.02  5.99  0.05 -4.02
  1.77  0.12  2.53 -1.77 -4.10  0.05 -6.19  4.10
  0.12  1.94 -1.77 -0.12  0.05 -4.02  4.10 -0.05
  0.68  1.66 -0.63 -0.68  1.08  1.46  0.17 -1.08
  1.66  3.17 -0.68 -1.66  1.46  3.27 -1.08 -1.46
-0.63  0.68 -0.27  0.63  0.17 -1.08  1.33 -0.17
-0.68 -1.66  0.63  0.68 -1.08 -1.46 -0.17  1.08
%disp(num2str(C4,fmtstr))
0.00  2.00 -2.00 -0.00 -0.00  2.00 -2.00  0.00
  2.00  3.00 -0.00 -2.00  2.00  3.00 -0.00 -2.00
-2.00  0.00 -3.00  2.00 -2.00 -0.00 -3.00  2.00
  0.00 -2.00  2.00 -0.00 -0.00 -2.00  2.00  0.00
  0.00 -2.00  2.00 -0.00  0.00  4.00 -4.00  0.00
-2.00 -3.00  0.00  2.00  4.00  6.00  0.00 -4.00
  2.00  0.00  3.00 -2.00 -4.00  0.00 -6.00  4.00
-0.00  2.00 -2.00  0.00 -0.00 -4.00  4.00 -0.00
-0.00  2.00 -2.00  0.00  0.80  1.60 -0.40 -0.80
  2.00  3.00 -0.00 -2.00  1.60  3.20 -0.80 -1.60
-2.00 -0.00 -3.00  2.00 -0.40 -0.80  0.20  0.40
  0.00 -2.00  2.00 -0.00 -0.80 -1.60  0.40  0.80
```
```matlab
»disp(num2str(C5,fmtstr))
0.00 2.00 -2.00 -0.00 -0.00 2.00 -2.00 0.00
2.00 3.00 -0.00 -2.00 2.00 3.00 -0.00 -2.00
-2.00 0.00 -3.00 2.00 -2.00 -0.00 -3.00 2.00
0.00 -2.00 2.00 -0.00 -0.00 -2.00 2.00 0.00
0.00 -2.00 2.00 -0.00 0.00 4.00 -4.00 0.00
-2.00 -3.00 0.00 2.00 4.00 6.00 0.00 -4.00
2.00 0.00 3.00 -2.00 -4.00 0.00 -6.00 4.00
-0.00 2.00 -2.00 0.00 -0.00 -4.00 4.00 -0.00
-0.00 2.00 -2.00 0.00 1.00 2.00 -0.00 -0.00
2.00 3.00 -0.00 -2.00 2.00 4.00 -0.00 -0.00
-2.00 -0.00 -3.00 2.00 0.00 -0.00 1.00 2.00
0.00 -2.00 2.00 -0.00 -0.00 -0.00 2.00 4.00

Original image(matrix)

»disp(num2str(C,fmtstr))
0.00 2.00 -2.00 0.00 0.00 2.00 -2.00 0.00
2.00 3.00 0.00 -2.00 2.00 3.00 0.00 -2.00
-2.00 0.00 -3.00 2.00 -2.00 0.00 -3.00 2.00
0.00 -2.00 2.00 0.00 0.00 -2.00 2.00 0.00
-0.00 -2.00 2.00 -0.00 0.00 4.00 -4.00 0.00
-2.00 -3.00 -0.00 2.00 4.00 6.00 0.00 -4.00
2.00 -0.00 3.00 -2.00 -4.00 0.00 -6.00 4.00
-0.00 2.00 -2.00 -0.00 0.00 -4.00 4.00 0.00
0.00 2.00 -2.00 0.00 1.00 2.00 0.00 0.00
2.00 3.00 0.00 -2.00 2.00 4.00 0.00 0.00
-2.00 0.00 -3.00 2.00 0.00 0.00 1.00 2.00
0.00 -2.00 2.00 0.00 0.00 0.00 2.00 4.00
```
clear
B = [1 2;2 4];
A = kron(eye(2),B) - kron(B',eye(2));
C = [A,A;-A,2*A;A,kron(eye(2),B)];
[U,S,V] = svd(C);
svals = diag(S);
disp(svals);
C1 = svals(1)*U(:,1)*V(:,1)';
C2 = C1 + svals(2)*U(:,2)*V(:,2)';
C3 = C2 + svals(3)*U(:,3)*V(:,3)';
C4 = C3 + svals(4)*U(:,4)*V(:,4)';
C5 = C4 + svals(5)*U(:,5)*V(:,5)';
discrep = zeros(5,1);
discrep(1) = norm(C-C1,'fro');
discrep(2) = norm(C-C2,'fro');
discrep(3) = norm(C-C3,'fro');
discrep(4) = norm(C-C4,'fro');
discrep(5) = norm(C-C5,'fro');
plot(1:5, discrep, 'ko', 1:5, discrep, 'k');
title('Frobenius Norm Discrepancy', 'FontSize', 14);
pause
mesh(C1); title('C1', 'FontSize', 14)
pause
mesh(C2); title('C2', 'FontSize', 14)
pause
mesh(C3); title('C3', 'FontSize', 14)
pause
mesh(C4); title('C4', 'FontSize', 14)
pause
mesh(C5); title('C5', 'FontSize', 14)
imagesc(C*10) % scaled image of original matrix

imagesc(C1*10) % include one term in sum
Two terms

Three terms
Four terms

Agrees with first image of entire picture(matrix).

Five terms

12 7.4