1. (25 points) Evaluate the circulation \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) using Stokes’s theorem. Here the vector field \( \mathbf{F} = x^2\mathbf{i} - xz\mathbf{j} + 5y^2\mathbf{k} \) and the path \( C \) is the circle in the plane \( x = 4 \) defined by equations: \( y^2 + z^2 = 2, x = 4 \). You must use Stokes’s theorem to do the problem. Any other method will not be accepted.

**Solution:** By Stokes’s theorem \( \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A} \), where \( S \) is the circular disc bounded by \( C \) in the plane \( x = 4 \). Since \( S \) is in the plane \( x = 4 \) the unit surface normal \( \mathbf{n} \) is a constant vector which can be chosen as \( \mathbf{i} \). Now a direct calculation gives

\[
\nabla \times \mathbf{F} = \begin{vmatrix}
    \mathbf{i} & \mathbf{j} & \mathbf{k} \\
    \partial_x & \partial_y & \partial_z \\
    x^2 & -xz & 5y^2
\end{vmatrix} = (10y + x)\mathbf{i} - z\mathbf{k}.
\]

We plug these expressions in the above integral to get

\[
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A} = \iiint_R ((10y + x)\mathbf{i} - z\mathbf{k}) \cdot (10y + x) dA = \iiint_R (10y + 4) dA,
\]

where region \( R \) is the disc \( y^2 + z^2 \leq 2 \) in the \( y-z \) plane. Look before we leap when it comes to evaluating the above integral. The integral \( \iiint_R y dA \) is zero because of symmetry of integrand \( y \) and region \( R \). Hence \( \iiint_R (10y + 4) dA = 10 \iiint_R y dA + \iiint_R 4 dA = 0 + 4 \text{(area of } R) = 4\pi (\sqrt{2})^2 = 8\pi. \)

2. (25 points) A circular cable of infinite length aligned with the \( z \)-axis is equipped with a constant charge density \( c \). The radius of the cable is \( a \). Calculate the electric field \( \mathbf{E} \) in the region inside the cable. Gauss’s law must be applied to solve the problem.

**Solution:** Choose a field point \( P \) where the distance between \( P \) and the axis of the cable is \( r, r < a \). The Gaussian surface \( S \) is selected as the closed cylindrical surface with radius \( r \) and length \( L \). By Gauss’ law we have

\[
\iint_S \mathbf{D} \cdot d\mathbf{A} = Q_{\text{total}}.
\]

To evaluate the flux integral \( \iint_S \mathbf{D} \cdot d\mathbf{A} \) we decompose it as

\[
\iint_S \mathbf{D} \cdot d\mathbf{A} = \iint_{\text{upper lid}} (*) dA + \iint_{\text{lower lid}} (*) dA + \iint_{\text{lateral face}} (*) dA = I_1 + I_2 + I_3.
\]

To calculate \( I_1 \), use the fact that the surface normal \( \mathbf{n} \) of the lid is \( \mathbf{k} \) and \( \mathbf{D} \) is radial so that they are perpendicular. Hence \( \mathbf{D} \cdot \mathbf{n} = 0 \). This gives \( I_1 = 0 \). Similarly \( I_2 = 0 \). Now since \( \mathbf{D} \) and \( \mathbf{n} \) are both radial on the lateral face of \( S \), hence \( \mathbf{D} \cdot \mathbf{n} = |\mathbf{D}| \) and \( I_3 = \int_{\text{lateral face}} |\mathbf{D}| dA. \) Again \( |\mathbf{D}| \) is everywhere the same on the lateral face, it is a constant. Hence \( \int_{\text{lateral face}} |\mathbf{D}| dA = |\mathbf{D}| \int_{\text{lateral face}} dA = |\mathbf{D}| \text{(area of the lateral face)} = |\mathbf{D}| (2\pi r L) \). We collect the result:

\[
\iint_S \mathbf{D} \cdot d\mathbf{A} = |\mathbf{D}| (2\pi r L).
\]

To calculate \( Q_{\text{total}} \), we note that the solid \( B \) enclosed in the Gaussian surface \( S \) is a cylinder of radius \( r \) and length \( L \). It is an simple fact that the volume on \( B \) \( = (\pi r^2) L \). Hence the total charge contained in \( S \) is just the constant charge density \( c \) times \( \text{volume of } B \Rightarrow Q_{\text{total}} = c(\pi r^2)L \)

Now we equate \( |\mathbf{D}| (2\pi r L) = c(\pi r^2)L \Rightarrow |\mathbf{D}| = \frac{c(\pi r^2)L}{(2\pi r L)} = \frac{c}{2} r \Rightarrow |\mathbf{E}| = \frac{c}{2\epsilon} r \). We summarize:
\[ E = \frac{c}{2\pi} r a, r < a \]