1. (25 points) Find $\text{Re}((-1)^{1+i})$ (Hint: it is multiple-valued)

(Solution: By the definition of complex power we have $(-1)^{1+i} = e^{(1+i)\ln(-1)}$. Now $\ln(-1) = \ln 1 + i(2k + 1)\pi = i(2k + 1)\pi$. Plug this in the exponent of the complex power we get $e^{(1+i)i(2k+1)\pi} = e^{-i(2k+1)\pi + i(2k+1)\pi} = e^{-i(2k+1)\pi} e^{i(2k+1)\pi}$

$= e^{-i(2k+1)\pi}(\cos(2k + 1)\pi + i \sin(2k + 1)\pi)$. Since $\cos(2k + 1)\pi = -1$ and $\sin(2k + 1)\pi = 0$, we arrive at $(-1)^{1+i} = -e^{-i(2k+1)\pi}$. The answer is $\text{Re}((-1)^{1+i}) = -e^{-i(2k+1)\pi}$, where $k$ is an integer.)

2. (25 points) Find three values of $z$ so that $\cos z = i$.

(Solution: $\cos z = \frac{e^{iz} + e^{-iz}}{2} = i$ $\Rightarrow e^{iz} + e^{-iz} = 2i$ $\Rightarrow e^{2iz} - 2ie^{iz} + 1 = 0$. Now let $X = e^{iz}$, the previous equation is transformed as $X^2 - 2iX + 1 = 0$. To solve the equation we invoke the quadratic formula $\Rightarrow X = (1 \pm \sqrt{2})i$. Finally we need to solve $e^{iz} = (1 \pm \sqrt{2})i$ for $z$. Taking complex logarithm leads to $iz = \ln((1 \pm \sqrt{2})i) \Rightarrow z = -i \ln((1 \pm \sqrt{2})i)$. The logarithm here is the multiple-valued complex logarithm. By definition $\ln((1 + \sqrt{2})i) = \ln(1 + \sqrt{2}) + i\left(\frac{\pi}{2} + 2k\pi\right)$, and $\ln((1 - \sqrt{2})i) = \ln(\sqrt{2} - 1) + i\left(-\frac{\pi}{2} + 2m\pi\right)$

The solution $z = \left(\frac{\pi}{2} + 2k\pi\right) - i \ln(1 + \sqrt{2})$ or $z = \left(-\frac{\pi}{2} + 2m\pi\right) - i \ln(\sqrt{2} - 1)$, where $k$ and $m$ are integers. This exhausts all solutions to the equation $\cos z = i$. To get three solutions we may specialize $z$ by let $k = 0, 1, 2$ for instance. Thus $z = \frac{\pi}{2} - i \ln(1 + \sqrt{2})$, $\frac{5\pi}{2} - i \ln(1 + \sqrt{2})$, $\frac{\pi}{2} - i \ln(1 + \sqrt{2})$.

Any other choice for three values of $k$ and $m$ are equally good.)