Approximation Algorithms

Department of Mathematics and Computer Science
Drexel University

Primal-Dual Method:

- Remember from last lecture the weighted vertex-cover problem:
  - Given: $G=(V,E)$ and weights $w_1,\ldots,w_n$ on $V$
  - Task: Find a subset of vertices $S$ satisfying:

$$\min_{S \subseteq V} \sum_{i \in S} w_i$$

- subject to:

$$\forall (i, j) \in E, i \in S \lor j \in S$$
LP Relaxation

- We saw an algorithm [Hochbaum] that converted Vertex Cover to an Integer Programming (IP) problem, relaxed the IP to a linear programming (LP) problem, and then used rounding to convert the LP solution back to integers:

  Variables: $x_1, ..., x_n$

  $x_i = 1 \Rightarrow$ vertex $i \in S$

  Objective:

  $\min \sum_{i=1}^{n} w_i x_i$

  Constraints: $x_i + x_j \geq 1, \forall (i, j) \in E$

  $0 \leq x_i \leq 1$

Example

- To round, we take the optimal LP solution, $x^*$, and choose as our solution to the vertex cover

  $VC = \{ i : x_i \geq 0.5 \}$

- But this approach has two problems.
  - First, solving the LP is expensive, so we'd like to find something cheaper.
  - Second, for some algorithms rounding may not work as a method for converting the LP optimum back into an integer solution that has provably good approximation bounds.

- The Primal-Dual method is a technique that can be used across a broad class of problems to quickly find an integer solution with provable approximation bounds.
Dual LP:

For any LP there is a dual LP:

\[
\begin{align*}
\min \quad & c^t x \\
\text{S.T.} \quad & Ax \geq b \\
\quad & x \geq 0
\end{align*}
\quad \iff 
\begin{align*}
\max \quad & b^t y \\
\text{S.T.} \quad & A^t y \leq c \\
\quad & y \geq 0
\end{align*}
\]

For example:

The Primal-Dual pair for weighted vertex-cover will be:

\[
\begin{align*}
\min \quad & \sum_{i=1}^{n} w_i x_i \\
\text{S.T.} \quad & x_i + x_j \geq 1, \forall (i, j) \in E \\
& 0 \leq x_i \leq 1
\end{align*}
\quad \iff 
\begin{align*}
\max \quad & \sum_{i=1}^{n} y_{ij} \\
\text{S.T.} \quad & \sum_{j: (i, j) \in E} y_{ij} \leq w_i, \forall i \in V \\
& y_{ij} \geq 0
\end{align*}
\]
Little more about LP:

- Given a LP, a point $x$ is said to be feasible if it satisfies all the constraints.

- LPs can be classified according to their set of feasible solutions:
  - **Feasible**: if a feasible solution exists:
    - **Bounded**: if a minimum exists
    - **Unbounded**: if we can find feasible solutions of arbitrary small values
  - **Infeasible**: if the set of feasible solutions is empty.

Important Theorem!

- For a linear system $Ax=b$, exactly one of the following holds:
  - Either
    $$\exists x, Ax = b$$
  - Or
    $$\exists y, A' y = 0 \text{ and } b' y = 1$$
Farkas Lemma

For any system of inequalities $Ax=b$, $x \geq 0$, exactly one of the following holds:

- Either
  $$\exists x \geq 0, Ax = b$$

- Or
  $$\exists y, A'y \geq 0 \text{ and } b'y \leq 0$$

Back to Primal Dual Pair

For the primal dual pair

$$\begin{align*}
\min & \quad c'x \\
\text{S.T.} & \quad Ax \geq b, \quad x \geq 0
\end{align*} \iff \begin{align*}
\max & \quad b'y \\
\text{S.T.} & \quad A'y \leq c, \quad y \geq 0
\end{align*}$$

Let $x^*$ denote the optimal solution of primal, and let $y$ be a feasible solution for dual, then

$$c'x^* \geq y'Ax^* = y'b = b'y,$$

Where the inequality follows since $x^* \geq 0$.

So the dual problem finds the best possible lower bound for primal!
Weak Duality

- Given feasible solutions $x$ and $y$ to the primal and dual problems, then
  \[ c^t x \geq b^t y \]

- In particular,
  \[ \min c^t x \geq \max b^t y \]

Proof:

- Given any solution $x$ and $y$ to the primal and dual respectively, we have: $x \geq 0, Ax - b \geq 0, y \geq 0$
- So, $y'(Ax - b) = y'Ax - y'b \geq 0$
- Likewise, $y \geq 0, b - A'y \geq 0, x \geq 0$
- Implying: $c^t x - y'Ax \geq 0$
- Finally, adding the two inequalities we will have
  \[ c^t x - y'b \geq 0 \]
For the WVC:

\[
\sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} \left( \sum_{j: (i,j) \in E} y_{ij} \right) x_i = \sum_{(i,j) \in E} (x_i + x_j) y_{ij} \geq \sum_{(i,j) \in E} y_{ij}
\]

In fact (strong duality):

- If a linear programming problem has an optimal solution \( x^* \), then its dual has an optimal solution \( y^* \), and

\[
c^T x^* = b^T y^*
\]
Primal-Dual Method for WVC

Instead of solving the LP or its Dual optimally something slightly different will be done.
Construct

- A vertex cover (i.e. an integral solution to the LP).
- A feasible solution to the Dual. (But not necessarily an optimal solution). such that

\[ \sum_{i=1}^{n} w_i x_i \leq 2 \sum_{(i,j) \in E} y_{ij} \leq 2LP_{(Dual)} = 2LP \leq 2OPT \]

A vertex cover (i.e. an integral solution to the LP).

Algorithm

Initially : let \( y = 0, x = 0 \)

this satisfies \( y_{ij} \geq 0 \) and \( \sum y_{ij} \leq w_j \)
but not \( x_i + x_j \geq 1 \), so it is not a VC

While \( x \) is not a vertex cover

choose \((i,j) \in E \) S.T. \( x_i + x_j = 0 \)

increase \( y_{ij} \) until wither \( \sum_{k, (i,j) \in E} y_{ik} = w_i \) or \( \sum_{k, (i,j) \in E} y_{kj} = w_j \)

i.e. increase \( y_{ij} \) until one constraint becomes tight

\( x_i = 1 \) if constraint is tight for \( i \)
else \( x_j = 1 \)
Observe that:

- $x$ is vertex cover.
- $y$ is dual feasible.
- We have: $\sum_{i=1}^{n} w_i x_i \leq 2\text{OPT}$

- At each step we kept the constraints tight $\forall i, w_i = \sum_{j(i,j) \in E} y_{ij}$
  \[ \Rightarrow \sum_{i=1}^{n} w_i x_i = \sum_{i=1}^{n} \sum_{j(i,j) \in E} x_i y_{ij}, \]
- But for each $i$, $x_i = 0$ or $1$:
  \[ \sum_{i=1}^{n} \sum_{j(i,j) \in E} x_i y_{ij} = \sum_{i=1}^{n} \sum_{j(i,j) \in E} y_{ij} \leq 2 \sum_{j(i,j) \in E} y_{ij} \leq 2 \max_{y} \sum_{j(i,j) \in E} y_{ij} = 2 \min_{i=1}^{n} \sum_{j(i,j) \in E} w_j x_i = 2\text{OPT} \]

The Generalized Steiner Tree Problem:

- We are given a graph $G=(V,E)$ with edge weights and a set of vertices selected out of $V$ say pairs $s_p$, $t_i$ such that $s_i$ and $t_i$ need to communicate. We'd like to find a minimum cost sub-graph such that all the $<s_p, t_i>$ pairs are connected.
- Given: $G=(V,E)$ w.l.o.g. fully connected,
  - weights $w_e >= 0$, for every edge $e$ in $E$
  - $<s_{1}, t_{1}>, <s_{2}, t_{2}>, \ldots, <s_{k}, t_{k}>$ pairs of communicating edges
- Find: A min-cost sub-graph $H$ of $G$ such that for all values of $i$ in $\{1, \ldots, k\}$ the pairs $<s_p, t_i>$'s are connected in $H$. 

Two Definitions:

- Define $L(S)$ as the indicator of subset $S$ of $V$ if $S$ forms a cut that separates one of the $<s_i, t_i>$ pairs. More formally:
  \[
  L(S) : 2^V \rightarrow \{0,1\} \\
  L(S) = 1 \iff \exists i : |S \cap \{s_i, t_i\}| = 1
  \]

- Also, for a graph $G=(V,E)$ and a subset of vertices $S$ define $d(S) = \{(i,j) \in E : i \in S \text{ and } j \not\in S\}$. That is, $d(S)$ is the set of edges that cross the cut boundary.

ILP Formulations:

Variables: 
\[
x_e = \begin{cases} 
1 & \text{if } e \in H \\
0 & \text{if } e \not\in H 
\end{cases} \quad \forall e \in E
\]

Objective: 
\[
\min \sum_{e \in E} w_e x_e
\]

Constraints: 
\[
\sum_{e \in \delta(S)} x_e \geq 1, \quad \forall S : L(S) = 1 \\
x_e \in \{0,1\}
\]
Relaxation:

- We will relax the integrality condition to \( x_e \geq 0 \). As a result:

\[
Opt_{\text{Steiner}} \geq \min_{x \geq 0} \sum_{e \in E} w_e x_e
\]

Dual Problem:

Objective: \[ \max_y \sum_{S \subseteq V : L(S) = 1} y_s \]

Constraints: \[ \sum_{s : e \in \delta(S)} y_s \leq w_e, \forall e \in E \]
\[ y_s \geq 0 \]
Primal-dual Algorithm:

- **Initialization:**
  - Let $F$ be the current solution (Eventually $F$ will be the solution $H$)
  - $F \leftarrow \emptyset$
  - $k \leftarrow 0$
  - $y \leftarrow 0$
  - $C$ be the connected components of $F$
    - $C = \{\{v_1\},\{v_2\},...,\{v_s\}\}$
  - $\forall i \in V$ Let $d(i) \leftarrow 0$ ($d(i)$ will represent $\sum_{s \in S} y_j$)

- We will start with each vertex in its connected component.
  - Then we will grow little balls around the components which disconnected communicators. When two balls merge, their connected components will be merged, add the edge between them, continue!

The algorithm

While $F$ is not feasible (i.e. $\exists C \in C : L(\overline{C}) = 1$)

// $\forall C \in C : L(\overline{C}) = 1$ we will increase $\gamma_C$ till some edge constarint becomes tight.

Find an edge $(i,j) \in E$ with $i \in C_p$ and $j \in C_q$ that minimizes $\epsilon = \frac{w_{ij} - (d(i) + d(j))}{L(C_p) + L(C_q)}$

Note that $d(i) + d(j) = \sum_{s(i,j) \in \delta(S)} y_j$

$k \leftarrow k + 1$

$e_s \leftarrow (i,j)$

$F \leftarrow F \cup e_s$

$\forall C \in C : L(\overline{C}) = 1$ increase $d(i)$ by $\epsilon$

$C \leftarrow C - \{C_p, C_q\} + \{C_p \cup C_q\}$

For $j \in k,...,1$ if $F \setminus \{e_j\}$ is feasible then $F \leftarrow F - e_j$

Print $F$
Example:

- Communicating pairs: <u,v>, <s,t>.

Correctness

- Given a Steiner tree problem, the above algorithm gives a feasible solution H and a feasible LP dual solution y such that:
  \[ \min_{x \geq 0} \sum_{e \in E} w_e x_e \leq \max_y \sum_{S \subseteq V} y_S \leq \text{OPT} \]
  \[ \leq \sum_{e \in H} w_e \leq 2 \sum_{S \subseteq V} y_S \leq 2 \max_y \sum_{S \subseteq V} y_S \leq 2 \text{OPT} \]