Limits and Continuity for the Trigonometric Functions
Thus all trigonometric functions are continuous in their natural domain.
One important tool for finding limits is the “squeezing theorem”

**Theorem.** Let $f$, $g$, and $h$ be functions satisfying

$$g(x) \leq f(x) \leq h(x)$$

for all $x$ in an open interval containing $c$, except possibly at $c$. If:

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

then

$$\lim_{x \to c} f(x) = L$$

We will not prove this result, but it is clearly illustrated by the following diagram.
The graph shows three functions: $h(x)$, $f(x)$, and $g(x)$. The point $(0,0)$ is labeled, indicating the origin. A vertical line at $x = c$ is also depicted.
Example. Compare $\sin(1/x)$ and $x\sin(1/x)$ at 0.

As $x$ tends to 0, $1/x$ tends to infinity. Thus, as $x$ tends toward 0, there are values of $x$ for which $1/x$ is $2\pi$, $4\pi$, $6\pi$, $8\pi$, etc. At all of these points, the sine is 0. There are also, as $x$ tends toward 0, values of $x$ for which $1/x$ is $\pi/2$, $(\pi/2) + 2\pi$, $(\pi/2) + 4\pi$, $(\pi/2) + 6\pi$, etc., and at these points the sine is 1. We conclude that the function $\sin(1/x)$ has no limit as $x$ tends to 0.
The function $x \sin(1/x)$ has the following graph.

This function is very difficult to analyze directly, but we can use the squeezing theorem. Since $-1 \leq \sin(x) \leq 1$ it follows that $-x \leq x \sin(x) \leq x$.

Since $\lim_{x \to 0} x = \lim_{x \to 0} -x = 0$ we have $\lim_{x \to 0} x \sin(1/x) = 0$.
Two Important Trigonometric Limits

Area of green sector is \( \frac{1}{2} \cdot r^2 \cdot x \)
\[ \text{Area} = \frac{\sin(x)}{2} \]

\[ \text{Area} = \frac{x}{2} \]

\[ \text{Area} = \frac{\tan(x)}{2} \]
Thus $\sin(x) < x < \tan(x)$, and so if we divide by $\sin(x)$, we have

$$1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$$

or, after taking reciprocals

$$\cos(x) < \frac{\sin(x)}{x} < 1 \quad (*)$$

for every $x$ between 0 and $\pi/2$. This also holds between $-\pi/2$ and 0, since $\sin(-x) = \sin(x)$ and $\cos(-x) = \cos(x)$.

Thus (*) holds in the interval $(-\pi/2, \pi/2)$ around 0.
Since \( \lim_{x \to 0} \cos(x) = \cos(0) = 1 \) the squeezing theorem shows that

\[
\lim_{x \to 0} \frac{\sin(x)}{x} = 1
\]

The second limit is found from this by using the identity

\[
\sin^2(x) = 1 - \cos^2(x)
\]

\[
\lim_{x \to 0} \frac{1 - \cos(x)}{x} = \lim_{x \to 0} \frac{1 - \cos(x) \cdot 1 + \cos(x)}{x(1 + \cos(x))} = \lim_{x \to 0} \frac{1 - \cos^2(x)}{x(1 + \cos(x))}
\]

\[
= \lim_{x \to 0} \frac{\sin^2(x)}{x(1 + \cos(x))} = \lim_{x \to 0} \frac{\sin(x) \cdot \sin(x)}{x(1 + \cos(x))} = 0
\]
Exercise. Find any points of discontinuity for the function

\[ f(x) = \frac{4}{1 - 2\cos(x)} \]

Solution. The function is continuous except where the denominator is 0. This happens when \( 1 - 2\cos(x) = 0 \), or \( \cos(x) = \frac{1}{2} \).

Thus \( x = \frac{\pi}{3}, -\frac{\pi}{3} \) or these angles plus any multiple of \( 2\pi \).

This is shown in the following picture.
Example. Compute \[ \lim_{x \to \infty} \sin \left( \frac{2}{x} \right) \]

Solution. Since the sine is a continuous function, we know by a previous theorem that

\[
\lim_{x \to \infty} \sin \left( \frac{2}{x} \right) = \sin \left( \lim_{x \to \infty} \frac{2}{x} \right) = \sin(0) = 0
\]
Example. Compute \( \lim_{h \to 0} \frac{\sin(h)}{5h} \)

Solution. Since constants can come out of the limit, we have

\[
\lim_{h \to 0} \frac{\sin(h)}{5h} = \frac{1}{5} \lim_{h \to 0} \frac{\sin(h)}{h} = \frac{1}{5}
\]

Example. Compute \( \lim_{s \to 0} \frac{\sin(5s)}{s} \)

Solution. We make a substitution, \( u = 5s \). Then

\[
\lim_{s \to 0} \frac{\sin(5s)}{s} = 5 \lim_{s \to 0} \frac{\sin(5s)}{5s} = 5 \lim_{u \to 0} \frac{\sin(u)}{u} = 5
\]
Example. Compute

$$\lim_{\theta \to 0} \left( \frac{\theta^2}{1 - \cos \theta} \right)$$

Solution. This expression must be changed to be understood. We will multiply the top and bottom by $1 + \cos(\theta)$

$$\lim_{\theta \to 0} \left( \frac{\theta^2}{1 - \cos \theta} \right) = \lim_{\theta \to 0} \left( \frac{\theta^2}{1 - \cos \theta} \right) \left( \frac{1 + \cos \theta}{1 + \cos \theta} \right)$$

$$= \lim_{\theta \to 0} \left( \frac{\theta^2 (1 + \cos \theta)}{1 - \cos^2 \theta} \right) = \lim_{\theta \to 0} (1 + \cos \theta) \left( \frac{\theta^2}{\sin^2 \theta} \right)$$

$$= \lim_{\theta \to 0} (1 + \cos \theta) \lim_{\theta \to 0} \left( \frac{\theta^2}{\sin^2 \theta} \right) = 2$$
Example. Compute \[ \lim_{h \to 0} \left( \frac{h}{\tan(h)} \right) \]

Solution. Here we begin with a trigonometric identity.

\[
\lim_{h \to 0} \left( \frac{h}{\tan(h)} \right) = \lim_{h \to 0} \left( \frac{h}{\sin(h)/\cos(h)} \right) = \lim_{h \to 0} \left( \frac{h\cos(h)}{\sin(h)} \right)
\]

\[
= \left( \lim_{h \to 0} \cos(h) \right) \left( \lim_{h \to 0} \frac{h}{\sin(h)} \right) = 1
\]
Example. Compute \[
\lim_{t \to 0} \left( \frac{t+3\sin t}{t} \right)
\]

Solution. Here we begin by breaking up a sum.

\[
\lim_{t \to 0} \left( \frac{t+3\sin t}{t} \right) = \left( \lim_{t \to 0} \frac{t}{t} \right) + \left( \lim_{t \to 0} \frac{3\sin t}{t} \right)
\]

\[
= 1 + 3 \left( \lim_{t \to 0} \frac{\sin t}{t} \right) = 4
\]