Techniques of Differentiation

Recall that if $f$ is a function, and $y = f(x)$, then we can represent the derivative function by the following notations:

$$f'(x) \quad f' \quad \frac{d}{dx}[f(x)]$$

**Theorem 1.** The derivative of a constant function is 0, that is:

$$\frac{d}{dx}[c] = 0$$

**Proof.**

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c-c}{h} = \lim_{h \to 0} 0 = 0$$
**Theorem.** For any positive integer $n$,  
\[
\frac{d}{dx}[x^n] = nx^{n-1} \quad \text{or} \quad (x^n)' = nx^{n-1}
\]

**Proof.** The key is to have a reasonable expression for $(x + h)^n$.

\[
(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + hn
\]

All terms after the first have $h$ as a factor.
Then \[ \frac{f(x+h)-f(x)}{h} = \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2} x^{n-2}h^2 + \cdots + h^n - x^n}{h} \]

\[ = nx^{n-1} + \frac{n(n-1)}{2} x^{n-2}h + \cdots + h^{n-1} = nx^{n-1} + h \left[ \text{polynomial in } h \right] \]

Thus,

\[ \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} nx^{n-1} + h \left[ \text{polynomial in } h \right] \]

\[ = nx^{n-1} \]
Theorem. If $f$ has a derivative at $x$, and $c$ is a constant, then
\[ \frac{d}{dx}[cf(x)] = c \frac{d}{dx}[f(x)] \quad \text{or} \quad (cf)' = cf' \]

Proof. \[ \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} = \lim_{h \to 0} c \frac{f(x+h) - f(x)}{h} \]
\[ c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = cf'(x) \]

Example. Find the derivative of $4x^7$

Solution. Using the last two formulas, we have
\[ \left[ 4x^7 \right]' = 4 \left[ x^7 \right]' = 4(7x^6) = 28x^6 \]
**Theorem.** If $f$ and $g$ have a derivatives at $x$, then

\[
\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]
\]

or

\[(f \pm g)' = f' \pm g'
\]

**Proof.** First we consider the sum.

\[
\lim_{h \to 0} \frac{(f + g)(x+h) - (f + g)(x)}{h} = \lim_{h \to 0} \frac{f(x+h) + g(x+h) - [f(x) + g(x)]}{h}
\]

\[
= \lim_{h \to 0} \left[ \frac{f(x+h) - f(x)}{h} \right] + \lim_{h \to 0} \left[ \frac{g(x+h) - g(x)}{h} \right] = f'(x) + g'(x)
\]
For the difference, we combine the first part with a previous theorem.

\[
\frac{d}{dx} \left[ f(x) - g(x) \right] = \frac{d}{dx} \left[ f(x) + (-g(x)) \right] = \frac{d}{dx} \left[ f(x) \right] + \frac{d}{dx} \left[ -g(x) \right]
\]

\[
= \frac{d}{dx} \left[ f(x) \right] - \frac{d}{dx} \left[ g(x) \right]
\]

**Example.** Find the derivative of \( p(x) = 3 + 2x - x^4 + 3x^8 \)

**Solution.**

\[
\left[ p(x) \right]' = \left[ 3 \right]' + \left[ 2x \right]' + \left[ -x^4 \right]' + \left[ 3x^8 \right]' = 0 + 2\left[ x \right]' - \left[ x^4 \right]' + 3\left[ x^8 \right]'
\]

\[
= 2 - 4x^3 + 3(8x^7) = 2 - 4x^3 + 24x^7
\]
The product formula

It is not true that the derivative of the product is the product of the derivatives, or that a similar formula holds for the quotient. The correct formulas are more complicated.

**Theorem. (the product formula)** If $f$ and $g$ have a derivatives at $x$, then

$$
\frac{d}{dx} [f(x)g(x)] = f(x) \frac{d}{dx}[g(x)] + g(x) \frac{d}{dx}[f(x)]
$$

or

$$(fg)' = fg' + gf'$$
Proof.

\[
\lim_{h \to 0} \frac{(fg)(x+h)-(fg)(x)}{h} = \lim_{h \to 0} \left[ \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \right]
\]

\[
= \lim_{h \to 0} \left[ \frac{f(x+h)g(x+h)-f(x+h)g(x)}{h} \right] + \lim_{h \to 0} \left[ \frac{f(x+h)g(x)-f(x)g(x)}{h} \right]
\]

\[
= \lim_{h \to 0} f(x+h) \lim_{h \to 0} \left[ \frac{g(x+h)-g(x)}{h} \right] + \lim_{h \to 0} g(x) \lim_{h \to 0} \left[ \frac{f(x+h)-f(x)}{h} \right]
\]

\[
= f(x) \frac{d}{dx} \left[ g(x) \right] + g(x) \frac{d}{dx} \left[ f(x) \right]
\]
The following theorem shows the situation for the quotient of two functions. We will not go over the proof.

**Theorem. (the quotient formula)** If $f$ and $g$ have a derivatives at $x$, then

$$
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx} [f(x)] - f(x) \frac{d}{dx} [g(x)]}{g(x)^2}
$$

or

$$
\left( \frac{f}{g} \right)' = \frac{gf' - fg'}{g^2}
$$
Example. Find the derivative of the function $x^{-n}$, where $n$ is a positive integer.

Solution. \[ x^{-n} = \frac{1}{x^n} \]

Thus we can use the quotient rule to find the derivative.

\[
(x^{-n})' = \left( \frac{1}{x^n} \right)' = \frac{x^n(1)' - 1(x^n)'}{(x^n)^2} = \frac{-n(x^{n-1})}{x^{2n}} = \frac{-n}{x^{n+1}} = -nx^{-n-1}
\]

We see that the same rule holds for negative integer powers (and 0 powers) as for positive integer powers. Thus we have:

Theorem. For any integer $n$, positive, negative, or 0,

\[
\frac{d}{dx}[x^n] = nx^{n-1} \quad \text{or} \quad (x^n)' = nx^{n-1}
\]
Example. Find the derivative of the reciprocal of a function. That is, find \( \left( \frac{1}{g(x)} \right)' \)

Solution. \( \left( \frac{1}{g} \right)' = \frac{g'g - g'g}{g^2} = \frac{-g'}{g^2} \)

Example. Find the derivative of \( f(x) = (x^2 - 3)(2x^3 + x) \)

Solution 1.

\[ f'(x) = 2x^5 + x^3 - 6x^3 - 3x = 2x^5 - 5x^3 - 3x \]

Then the derivative is \( f'(x) = 2(5x^4) - 5(3x^2) - 3 = 10x^4 - 15x^2 - 3 \)
Solution 2. Using the product formula, we have

\[ f'(x) = (x^2 - 3)(2x^3 + x)' + (x^2 - 3)'(2x^3 + x) \]

\[ = (x^2 - 3)(6x^2 + 1) + (2x)(2x^3 + x) \]

\[ = (6x^4 + x^2 - 18x^2 - 3) + (4x^4 + 2x^2) \]

\[ = 10x^4 - 15x^2 - 3 \]

Example. Find the derivative of \( f(x) = \sqrt{x} \)

Solution.

\[ f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \]

\[ = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}} \]
Remark. \( \sqrt{x} = x^{1/2} \)

We showed that the derivative of this function is \( \frac{1}{2\sqrt{x}} \)

Thus we see that the derivative of \( x^{1/2} \) is \( \frac{1}{2}x^{-1/2} \)

It follows that the power formula

\[
(x^n)' = nx^{n-1}
\]

also works for \( n = \frac{1}{2} \). Later on we will show that it works for all fractions, and indeed for all real numbers.
Example. Find the derivative of $f(x)=\frac{1}{x}+\frac{1}{x^2}(3x^3+2)$

Solution.

$$f'(x)=\left(\frac{1}{x}+\frac{1}{x^2}\right)(3x^3+2)'+\left(\frac{1}{x}+\frac{1}{x^2}\right)'(3x^3+2)$$

$$=\left(\frac{1}{x}+\frac{1}{x^2}\right)(9x^2)+\left(\frac{-1}{x^2}+\frac{-2}{x^3}\right)'(3x^3+2)$$

$$=\left(\frac{1}{x}+\frac{1}{x^2}\right)(9x^2)+\left(\frac{-1}{x^2}+\frac{-2}{x^3}\right)(3x^3+2)$$

$$=\left(\frac{1}{x}+\frac{1}{x^2}\right)(9x^2)-\left(\frac{1}{x^2}+\frac{2}{x^3}\right)(3x^3+2)=6x+3-\frac{2}{x^2}-\frac{4}{x^3}$$
Example. Find the derivative of \( \frac{3}{\sqrt{x} + 2} \)

Solution. By the quotient formula

\[
\left( \frac{3}{\sqrt{x} + 2} \right)' = \frac{(\sqrt{x} + 2)' - 3(\sqrt{x} + 2)'}{(\sqrt{x} + 2)^2}
\]

\[
= -3 \left( \frac{1}{2\sqrt{x}} \right) = -3 \left( \frac{1}{2\sqrt{x}} \right) = -3
\]

\[
= \frac{-3}{(\sqrt{x} + 2)^2} = \frac{-3}{(x + 4\sqrt{x} + 4)} = \frac{-3}{(2\sqrt{x})(x + 4\sqrt{x} + 4)}
\]
Example. Find the derivative of \( \frac{4x+1}{x^2 - 5} \)

Solution. By the quotient formula

\[
\left( \frac{4x+1}{x^2 - 5} \right)' = \frac{(x^2 - 5)(4x+1)' - (4x+1)(x^2 - 5)'}{(x^2 - 5)^2}
\]

\[
= \frac{4(x^2 - 5) - (4x+1)(2x)}{x^4 - 10x^2 + 25}
\]

\[
= \frac{-4x^2 - 2x - 20}{x^4 - 10x^2 + 25}
\]