Inverse Functions

Suppose that \( f \) is a function. Recall that the domain of \( f \) (in the cases we are studying) is a set of real numbers - the allowable inputs - and the range of \( f \) is the set of all corresponding outputs.
**Question.** Is it possible to find a function $g$ that reverses the effect of $f$, that is, $g$ takes outputs $f(x)$ as inputs and returns the original input $x$ as its result. If so, the new function would have the range of $f$ as its domain and the domain of $f$ as its range, and symbolically, it would satisfy the equation

$$g(f(x)) = x \text{ for all } x \text{ in } \text{Dom}(f)$$

That is

$$(g \circ f)(x) = x \text{ for all } x \text{ in } \text{Dom}(f)$$

**Diagram:**

- Range of $f$ is the Domain of $g$.
- Domain of $f$ is the Range of $g$. 

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Example. Let $f(x) = x^3$.

The function $f$ takes any real number $x$ and cubes it. To get $x$ back from the output $x^3$, we must take the cube root.

Thus if we define $g$ by the formula $g(x) = \sqrt[3]{x}$ we have

$$(g \circ f)(x) = g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

so $g$ reverses the effect of $f$. Note also that

$$(f \circ g)(x) = f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

so $f$ also “inverts” or reverses the effect of $g$.

We are therefore led to the following definition of inverse function.
**Definition.** A function $f$ has an inverse $g$ if

$$(g \circ f)(x) = g(f(x)) = x \text{ for all } x \text{ in } \text{Dom}(f).$$

$$(f \circ g)(x) = f(g(x)) = x \text{ for all } x \text{ in } \text{Dom}(g).$$

It is clear that if $f$ has an inverse, then there can only be one, and that if $g$ is the inverse of $f$, then $f$ is the inverse of $g$. Thus invertible functions come in pairs.

We then write $g = f^{-1}$, and $f = g^{-1}$. 
Example. Determine if the following pairs of functions are inverses of one another.

(a) \( f(x) = 3x+4 \) \( g(x) = \frac{x-4}{3} \)

(b) \( f(x) = \sqrt[3]{x-1} \) \( g(x) = x^3+1 \)

(c) \( f(x) = x^2 \) \( g(x) = \sqrt{x} \)

Solution. (a) \[ f(g(x)) = 3 \left( \frac{x-4}{3} \right) + 4 = x \], \( g(f(x)) = \frac{(3x+4)-4}{3} = x \)

Both equations hold for all real \( x \), so the functions are inverses.
(b) \( f(g(x)) = \sqrt[3]{(x^3+1)}-1 = x \quad g(f(x)) = \left( (\frac{3}{\sqrt{x-1}})^3 + 1 \right) = x \)

Both equations hold for all real \( x \), so the functions are inverses.

(c) \( g(f(x)) = \sqrt{x^2} = |x| \)

This will not equal \( x \) if \( x \) is negative, so \( g \) is not the inverse of \( f \).
The above diagram shows the sort of thing that can make an inverse function impossible. If two (or more) inputs produce that same output, then there can be no inverse, since in this case we would have $x_1 = f^{-1}(y)$ and $x_2 = f^{-1}(y)$. Then $x_1$ and $x_2$ would be equal, contrary to assumption.
This leads us to the following definition.

**Definition.** A function $f$ is said to be one-to-one, if different inputs always produce different outputs.

It turns out that this difficulty is the only one that stands in the way of having an inverse. Thus we have the following result.

**Theorem.** A function $f$ has an inverse if and only if $f$ is one to one. The inverse is then denoted by $f^{-1}$, and $\text{Dom}(f^{-1}) = \text{Ran}(f)$ and $\text{Ran}(f^{-1}) = \text{Dom}(f)$. 
Graphical Meaning of One-to-one

The value (output) of the function $f(x)$ corresponds to the height of the graph of $f$. We know that the function $f$ does not have an inverse (is not one to one) if there are two distinct points $x_1$ and $x_2$ with $f(x_1) = f(x_2) = c$. In this case the horizontal line $y = c$ intersects the graph of $f$ in two places. Thus we have the convenient **horizontal line test** for having an inverse:

A function $f$ has an inverse (is one-to-one) if and only if no horizontal line intersects the graph of $f$ in more than one point.
**Example.** The function $f(x) = x^2$ does not have an inverse. It fails the horizontal line test for $y = 4$, as shown below (and for many other horizontal lines).

This function is clearly not one-to-one, because

\[
\begin{align*}
  f(-2) &= f(2) = 4 \\
  f(-7) &= f(7) = 49 \\
  f(-3) &= f(3) = 9
\end{align*}
\]

and in general, every output other than 0 has two inputs that produce it.
**Example.** The function $f(x) = x^3$ does have an inverse. Each horizontal line intersects its graph in exactly one point, since each real number has exactly one real cube root.

The function therefore has an inverse, and we have shown previously that this is the function $f^{-1}(x) = \sqrt[3]{x}$. 
**Example.** Let $f$ be the function whose formula is $f(x) = x^2$ and whose domain is defined to be all $x \geq 0$. This function is one-to-one, since no two different positive numbers can have the same square.

The graph is shown above. It clearly satisfies the horizontal line test. The above function has both domain and range equal to the set of all nonnegative real numbers, and so it has an inverse function with the same domain and range.
The inverse function $f^{-1}$ is shown below. Its domain and range also consist of all nonnegative numbers, and for any such number, the formula is $f^{-1}(x) = \sqrt{x}$

Thus we see that we can sometimes make a function one-to-one, and therefore invertible, by restricting its domain. All functions defined by odd powers of $x$ have inverses. Functions defined by even powers of $x$ have inverses if their domain is restricted to all non-negative numbers.
Sometimes it is difficult to tell from a formula, whether or not a function is one-to-one.

\[ x^3 - 3x + 2 \]

\[ x^3 - 3x^2 + 3x - 1 \]
Now to the second question; if $f(x)$ is the formula for the function $f$, and $f$ has an inverse, what is the formula $f^{-1}(x)$ for the inverse function?

If $f$ has an inverse, and $y = f(x)$, then $x = f^{-1}(y)$. This leads to a practical method for computing the inverse function.

1. Write the equation $y = f(x)$
2. Solve this equation for $x$ in terms of $y$. (getting $x = f^{-1}(y)$)
3. Substitute $x$ for $y$ in the formula $f^{-1}(y)$.
Example. Find the inverse of the function \( f(x) = 2x^5 \).

Solution.

1. The equation for \( f \) is \( y = 2x^5 \)

2. We solve to get

\[
x = \frac{5}{2} \sqrt[5]{y}
\]

3. We change variable and find that the inverse function is given by

\[
f^{-1}(x) = \frac{5}{2} \sqrt[5]{x}
\]
Example. Find the inverse of the function $f(x) = 3x - 7$

Solution.
1. The equation for $f$ is $y = 3x - 7$.

2. We solve to get $x = \frac{y+7}{3}$.

3. We change variable and find that the inverse function is given by the formula $f^{-1}(x) = g(x) = \frac{x+7}{3} = \frac{x}{3} + \frac{7}{3}$.

We can verify directly that $g$ is the inverse of $f$.

$$g(f(x)) = g(3x-7) = \frac{(3x-7)+7}{3} = x$$

$$f(g(x)) = f\left(\frac{x+7}{3}\right) = 3\left(\frac{x+7}{3}\right) - 7 = x$$
Example. Find the inverse of the function \( f(x) = \frac{x+1}{x-1} \)

Solution.
1. The equation for \( f \) is \( y = \frac{x+1}{x-1} \).

2. We solve for \( x \):
   \[
   yx - y = x + 1 \\
   yx - x = y + 1 \\
   x = \frac{y + 1}{y - 1}
   \]

3. We change variable and find that the inverse function is given by the formula
   \[
   f^{-1}(x) = \frac{x+1}{x-1} = f(x)
   \]

Thus \( f \) is its own inverse.
Example. Find the inverse of the function \( f(x) = \frac{5}{\sqrt{4x+2}} \)

Solution.
1. The equation for \( f \) is \( y = \frac{5}{\sqrt{4x+2}} \).

2. We solve for \( x \):
   \[
y^5 = 4x + 2
   \]
   \[
x = \frac{y^5 - 2}{4}
   \]

3. We change variable and find that the inverse function is given by the formula
   \[
f^{-1}(x) = \frac{x^5 - 2}{4}
   \]
The Graph of an inverse function

Suppose that $f$ is one to one (its graph satisfies the horizontal line test) and therefore it has an inverse $f^{-1}$. If the point $(a, b)$ is a point on the graph of $f$, then $b = f(a)$. This means that

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$ 

and so $(b, a)$ is a point on the graph of $f^{-1}$. We therefore have the following principle.

The points on the graph of $f^{-1}$ are found by reversing the coordinates of the points on the graph of $f$. Thus the graph of $f^{-1}$ is the reflection of the graph of $f$ in the line $y = x$. 

$y = x$
Example. The graph of $\sqrt[3]{x}$ is found by reflecting the graph of $y = x^3$ about the line $y = x$. 
**Example.** Let $f(x) = x^2$, for $0 \leq x < \infty$. Then $f^{-1}(x) = \sqrt{x}$ for $0 \leq x < \infty$. The graphs of the two functions, separately and together are shown below.
**Example.** Here is a preview of a topic to come later. The function \( \sin(x) \) is not one-to-one, as shown by its graph. We also show below the graph of a restriction of this function, the function given by:

\[
f(x) = \sin(x), \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}
\]

The function \( f \) is one to one, and so has an inverse.
The inverse of this function has the graph shown below. It is usually called $\sin^{-1}(x)$ or $\text{Arcsin}(x)$