Derivatives of Logarithmic and Exponential Functions

We begin by finding the derivative of a logarithmic function directly from the definition and the properties of logarithms. We remind you of the following facts:

1. \( \log_b \left( \frac{a}{c} \right) = \log_b a - \log_b c \)
2. \( \log_b (a^r) = r \log_b a \)
3. If \( f \) is a continuous function, and \( \lim_{x \to a} g(x) = g(a) \)
then \( \lim_{x \to a} f(g(x)) = f(g(a)) \)
4. \( e = \lim_{x \to \infty} \left( 1 + \frac{1}{x} \right)^x = \lim_{v \to 0} \left( 1 + v \right)^{\frac{1}{v}} \) where \( v = \frac{1}{x} \)
**Theorem.** The derivative of $f(x) = \log_b(x)$ is $(\log_b e)^{-1} x$

And more generally

$$\frac{d}{dx} \left[ \log_b(u) \right] = \log_b(e) \frac{1}{u} \frac{du}{dx}$$

**Proof.**

$$\frac{d}{dx} \left[ \log_b(x) \right] = \lim_{h \to 0} \frac{\log_b(x+h) - \log_b(x)}{h} \quad (\text{Definition})$$

$$= \lim_{h \to 0} \frac{1}{h} \log_b \left( \frac{x+h}{x} \right) \quad (\text{Fact 1})$$

$$= \lim_{h \to 0} \frac{1}{h} \log_b \left( 1 + \frac{h}{x} \right) \quad (\text{algebra})$$

$$= \lim_{v \to 0} \frac{1}{v} \log_b \left( 1 + \frac{h}{x} \right) \quad (\text{Substitute } v = h/x)$$
\[ \left[ \frac{1}{x} \right]_{\nu \rightarrow 0} \lim_{\nu} \frac{1}{\log_b (1+\nu)} \quad \text{(Constant 1/x comes outside limit)} \]

\[ \left[ \frac{1}{x} \right]_{\nu \rightarrow 0} \lim_{\nu} \log_b (1+\nu) \frac{1}{\nu} \quad \text{(Fact 2)} \]

\[ \left[ \frac{1}{x} \right] \log_b \left[ \lim_{\nu \rightarrow 0} (1+\nu) \frac{1}{\nu} \right] \quad \text{(Fact 3)} \]

\[ \left[ \frac{1}{x} \right] \log_b (e) \quad \text{(Fact 4)} \]
As a special case, when $b = e$, we have

**Theorem.** The derivative of $f(x) = \ln(x)$ is $\left(\log_e e\right) \frac{1}{x} = \frac{1}{x}$

It follows that $\frac{d[\ln(u)]}{dx} = \frac{1}{u} \frac{du}{dx}$

This simple formula explains in part the use of $e$ as a base for logarithms in mathematical work.

**Example.** Find the derivative of

(a) $\frac{\log(x)}{1+\log(x)}$  
(b) $\ln(\ln(x))$  
(c) $\ln(2+\sqrt{x})$
(a) \[
\left[ \frac{\log(x)}{1+\log(x)} \right]' = \left[ 1+\log(x) \right] \log(e) \frac{1}{x} - \log(x) \log(e) \frac{1}{x} \left[ 1+\log(x) \right]^2
\]
\[
= \frac{\log(e)\frac{1}{x}}{\left[ 1+\log(x) \right]^2} = \frac{\log(e)}{x[1+\log(x)]^2}
\]

(b) \[
\ln(\ln(x))' = \frac{1}{\ln(x)} \frac{d[\ln(x)]}{dx} = \frac{1}{x \ln(x)}
\]

(c) \[
\left[ \ln(2+\sqrt{x}) \right]' = \frac{1}{2+\sqrt{x}} \left( \frac{d[2+\sqrt{x}]}{dx} \right)
\]
\[
= \left( \frac{1}{2+\sqrt{x}} \right) \left( \frac{1}{2\sqrt{x}} \right) = \frac{1}{4\sqrt{x}+2x}
\]
Derivative of $\ln(x)$ is $\frac{1}{x}$.

Slope of tangent line is 1 at $x = 1$. 
Derivative of $\ln(x)$ is $\frac{1}{x}$.

Slope of tangent line at $x = 2$ is $1/2$. 
Derivative of \( \ln(x) \) is \( \frac{1}{x} \)

Slope of tangent line at \( x = 4 \) is \( \frac{1}{4} \).
If you are taking the derivative of the logarithm of a complicated expression involving primarily products, quotients, and powers, it is usually best to simplify the logarithm first before taking derivatives.

**Example.** Compute the derivative of the function

\[ f(x) = \ln \left( \frac{(x+1)^3 \cos(x)}{x\sqrt{2-x}} \right) \]

We can write this derivative as \( \frac{1}{u} \frac{du}{dx} \) but the computation of the derivative of \( u \) will be very long and involved.
**Better Solution:** First we simplify the function.

\[
\ln\left[ \frac{(x+1)^3 \cos(x)}{x\sqrt{2-x}} \right] = \ln[(x+1)^3] + \ln(\cos(x)) - \ln(x) - \ln(\sqrt{2-x})
\]

So \( f(x) = 3\ln(x+1) + \ln(\cos(x)) - \ln(x) - \frac{1}{2}\ln(2-x) \)

Then \( f'(x) = 3 - \frac{1}{x+1} + \frac{-\sin(x)}{\cos(x)} - \frac{1}{x} - \frac{1}{2(2-x)} \)

\[
= \frac{3}{x+1} - \tan(x) - \frac{1}{x} + \frac{1}{4-2x}
\]
Logarithmic Differentiation

Even if a function does not involve a logarithm, it’s differentiation can often be simplified by using a technique called logarithmic differentiation. This technique should be used on functions that are primarily composed of products, quotients, and roots.

**Example.** Find the derivative of the function

\[ y = \frac{\sin(x) \cos(x) \tan^3(x)}{\sqrt{x}} \]
1. We begin by taking the natural logarithm of both sides of the equation

\[ y = \frac{\sin(x) \cos(x) \tan^3(x)}{\sqrt{x}} \]

\[
\ln(y) = \ln \left( \frac{\sin(x) \cos(x) \tan^3(x)}{\sqrt{x}} \right) \\
= \ln(\sin(x)) + \ln(\cos(x)) + 3\ln(\tan(x)) - \frac{1}{2} \ln(x)
\]

2. Now take the derivative of both sides, treating \( y \) as an implicit function of \( x \) on the left, getting

\[
\frac{1}{y} \frac{dy}{dx} = \frac{\cos(x) - \sin(x)}{\sin(x) \cos(x)} + \frac{3 \sec^2(x) - 1}{\tan(x) \cos(x)} - \frac{1}{2x}
\]
Thus

\[
\frac{dy}{dx} = y \left[ \frac{\cos(x)}{\sin(x)} + \frac{-\sin(x)}{\cos(x)} + 3\frac{\sec^2(x)}{\tan(x)} - \frac{1}{2x} \right]
\]

= \[ y \left[ \cot(x) - \tan(x) + 3\frac{\sec^2(x)}{\tan(x)} - \frac{1}{2x} \right] \]

= \left[ \frac{\sin(x)\cos(x)\tan^3(x)}{\sqrt{x}} \right] \left[ \cot(x) - \tan(x) + 3\frac{\sec^2(x)}{\tan(x)} - \frac{1}{2x} \right]
Derivatives of arbitrary powers of $x$

**Theorem.** For any real number $r$, rational or irrational, we have

$$
\frac{d}{dx}\left[x^r\right] = rx^r - 1
$$

**Proof.** We use logarithmic differentiation. Let $y = x^r$. Then

$$
\ln(y) = \ln(x^r) = r\ln(x), \text{ therefore}
$$

$$
\frac{1}{y}\frac{dy}{dx} = r \quad \text{and so} \quad \frac{dy}{dx} = \frac{ry}{x} = \frac{rx^r}{x} = rx^r - 1
$$

**Examples.**

$$
\frac{d}{dx}\left[x^\pi\right] = \pi x^\pi - 1
$$

$$
\left[x^\sqrt{5}\right]' = \sqrt{5}x\sqrt{5} - 1
$$
Derivatives of Exponential Functions

We can also use logarithmic differentiation to find the
derivatives of the functions $f(x) = b^x$.

**Theorem.**

$$\left[ b^x \right]' = \ln(b) b^x$$
and, more generally,

$$\frac{d}{dx} \left[ b^u \right] = \ln(b) b^u \frac{du}{dx}$$

**Proof.** Let $y = b^x$. Then $\ln(y) = \ln(b^x) = x\ln(b)$. Therefore

$$\frac{1}{y} \frac{dy}{dx} = \ln(b) \quad \text{and} \quad \frac{dy}{dx} = y\ln(b) = \ln(b) b^x.$$ The second formula follows by the chain rule. If $b = e$, we have:

**Theorem.**

$$\left[ e^x \right]' = e^x$$
and, more generally,

$$\frac{d}{dx} \left[ e^u \right] = e^u \frac{du}{dx}$$
Derivative of $\exp(x)$ is $\exp(x)$

Slope of tangent line is 1 at $x = 1$

Height is 1 at $x = 1$. 
Derivative of $\exp(x)$ is $\exp(x)$
Derivative of $\exp(x)$ is $\exp(x)$

Slope of tangent line

Height
Example. Find the derivative of $e^{\cos(x)}$ with respect to $x$.

Solution:
\[
\left[ e^{\cos(x)} \right]' = e^{\cos(x)} [\cos(x)]' = -e^{\cos(x)} \sin(x)
\]

Example. Find the derivative of $2^{4x^3}$ with respect to $x$.

Solution:
\[
\frac{d}{dx} \left[ 2^{4x^3} \right] = \ln(2) 2^{4x^3} \left[ 4x^3 \right]' = \ln(2) \left( 12x^2 \right) 2^{4x^3}
\]
We have now shown the following rules for differentiation.

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Do not confuse 3 and 4 with 5
Miscellaneous Examples

Example. Find the derivative of $y = \ln(x^3)$ with respect to $x$.

Solution 1: \[
\frac{dy}{dx} = \frac{1}{x^3} \frac{d}{dx}[x^3] = 3x^2 = \frac{3}{x}
\]

Solution 2: $y = 3\ln(x)$ so \[
\frac{dy}{dx} = 3\frac{1}{x} = \frac{3}{x}
\]

Example. Find the derivative of $y = \ln(x)^3$ with respect to $x$.

Solution: \[
\frac{dy}{dx} = 3\ln(x)^2 \frac{d}{dx} \ln(x) = \frac{3\ln(x)^2}{x}
\]
Example. Find the derivative of $y = \sin^2(\ln(x))$ with respect to $x$.

Solution:
\[
\frac{dy}{dx} = 2\sin(\ln(x))\cos(\ln(x)) \frac{d}{dx} \ln(x) = \frac{2\sin(\ln(x))\cos(\ln(x))}{x}
\]

Example. Find the derivative of $y = \ln(\sin^2 x)$ with respect to $x$.

Solution:
\[
y = 2\ln(\sin(x)) \text{, so } \frac{dy}{dx} = \frac{2}{\sin(x)} \frac{d}{dx} \sin(x) = 2\frac{\cos(x)}{\sin(x)} = 2\cot(x)
\]
Example. Find the derivative of \( y = \ln \left( \frac{x^2 \sqrt{2-x}}{(x+3)^3(x-1)} \right) \) with respect to \( x \).

Solution: First simplify to get

\[
y = \ln(x^2) + \ln(\sqrt{2-x}) - \ln((x+3)^3) - \ln(x-1)
\]

\[
= 2\ln(x) + \frac{1}{2}\ln(2-x) - 3\ln(x+3) - \ln(x-1)
\]

Then

\[
y' = \frac{2}{x} + \frac{1}{2} \left( \frac{-1}{2-x} \right) \left( \frac{3}{x+3} - \frac{1}{x-1} \right) = \frac{2}{x} - \frac{1}{4-2x} - \frac{3}{x+3} - \frac{1}{x-1}
\]
Example. Find the derivative of \( y=e^{(x-e^{3x})} \) with respect to \( x \).

Solution: \[
\frac{dy}{dx}=e^{(x-e^{3x})} \frac{d}{dx}\left[x-e^{3x}\right]=e^{(x-e^{3x})}\left(1-3e^{3x}\right)
\]

Example. Find the derivative of \( y=\left(x^2+1\right)^{\sqrt{2}} \) with respect to \( x \).

Solution: \[
\frac{dy}{dx}=\sqrt{2}\left(x^2+1\right)^{\sqrt{2}-1} \frac{d}{dx}\left[x^2+1\right]=\sqrt{2}\left(x^2+1\right)^{\sqrt{2}-1}(2x)
\]
\[
=2\sqrt{2}x\left(x^2+1\right)^{\sqrt{2}-1}
\]
Example. Find the derivative of $\frac{5\sqrt{x-1}}{x+1}$ with respect to $x$ by using logarithmic differentiation.

Solution: Let $y = 5\sqrt{\frac{x-1}{x+1}}$.

Then $\ln(y) = \ln\left[5\frac{x-1}{x+1}\right] = \ln\left(\frac{x-1}{x+1}\right)$

so $\frac{1}{y} \frac{dy}{dx} = \frac{1}{5} \left[\frac{1}{x-1}\right] - \frac{1}{5} \left[\frac{1}{x+1}\right] = \frac{2}{5(x-1)(x+1)}$

$\frac{dy}{dx} = y \left[\frac{2}{5(x-1)(x+1)}\right] = \frac{5\sqrt{x-1}}{x+1} \left[\frac{2}{5(x-1)(x+1)}\right]$
Example. Find the derivative of $\ln(x)\tan(x)$ with respect to $x$ by using logarithmic differentiation.

Solution: Let $y = \ln(x)\tan(x)$

Then $\ln(y) = \ln\left[\ln(x)\tan(x)\right] = \tan(x)\ln(\ln(x))$

so $\frac{1}{y} \frac{dy}{dx} = \sec^2(x)\ln(\ln(x)) + \tan(x) \frac{1}{\ln(x)} \frac{1}{x}$

$\frac{dy}{dx} = y \left[ \sec^2(x)\ln(\ln(x)) + \tan(x) \frac{1}{\ln(x)} \frac{1}{x} \right]$