Example. You have 200 ft. of fence with which to fence off a rectangular plot of ground on the banks of a river. How should you arrange the fence in order to get the largest plot.
Let the dimensions of the plot be $x$ and $y$ with $y$ parallel to the river and $x$ perpendicular. Let $A$ be the area, and $P$ the perimeter of the plot. Then we know that

$$A = xy \quad \text{and} \quad P = 2x + y = 200$$

The first equation gives the target quantity (the one to be maximized or minimized) in terms of basic variables. The second equation is called the *constraint equation*, and gives a relationship among the basic variables.

The constraint equation (or equations) of a problem are used to write all the basic variables in terms of just one of them. Then by substituting into the equation for the target, we can write the target in terms of this single variable. Here we have $y = 200 - 2x$ and so we get the equation

$$A = x(200 - 2x) = 200x - 2x^2$$
The conditions of the example show that the largest possible value of $x$ is 100, and the smallest possible value is 0. Thus the problem become finding the absolute maximum of the function $A$ on the interval $[0, 100]$

$$\frac{dA}{dx} = 200 - 4x$$

This derivative is 0 when $x = 50$. Thus we need to test the values 0, 50, and 100 in the formula for area $A = x(200 - 2x) = 200x - 2x^2$

$$A(0) = 0 \quad A(50) = (50)(200 - 100) = 5000 \quad A(100) = 100(200 - 200) = 0$$

The maximum occurs when $x = 50$, ($y = 100$), and is 5000 square feet.
This suggests the following procedure.

**Step 1.** Draw an appropriate figure and label the quantities relevant to the problem.

**Step 2.** Find a formula for the quantity to be maximized or minimized.

**Step 3.** Using the constraint equations of the problem to eliminate variables, express the quantity to be maximized or minimized as a function of one variable.

**Step 4.** Find the interval of possible values for this variable from the physical restrictions in the problem.

**Step 5.** If applicable, use the techniques of the previous section to obtain the maximum or minimum.
Example. An open box is to be made from a 16 in. by 30 in. piece of cardboard by cutting out squares of equal size from the corners and bending up the sides. What should the size of the squares be so that the resulting box has a maximum volume?
Solution. Clearly the minimum value of $x$ is 0 and the maximum is 8. The volume $V$ of the resulting box is

$$V = x(16-2x)(30-2x) = 480x - 92x^2 + 4x^3$$

So we need the absolute maximum of this function on the interval $[0, 8]$.

$$\frac{dV}{dx} = 480 - 184x + 12x^2 = 4(120 - 46x + 3x^2)$$

This is 0 at $x = 12$ and $x = 10/3$ (using the quadratic formula).

The critical point $x = 12$ is outside of the interval $[0, 8]$, so we keep the point $10/3$ and add in the end points 0 and 8. The three values of $V$ that result are respectively

$$V(0) = 0 \quad V(10/3) = 19600/27 \approx 726 \quad V(8) = 0$$
Thus the maximum value occurs when $x = 10/3$. 
Example. A cylindrical can is made to hold 1 liter of oil. Find the dimensions of the can that will minimize the cost of the metal to manufacture the can.
Solution. In order to hold a constant volume, it is clear that if the height of the can increases, the radius of the base must decrease, and conversely. Thus in principle, either the radius or height can take any value from 0 to infinity.

Then let $V$ be the volume in cm$^3$ and $S$ the surface area in cm$^2$.

$$V = \pi r^2 h = 1000 \quad S = 2\pi r^2 + 2\pi rh$$

In this case, the first equation is the constraint equation. From it we can learn that

$$h = \frac{1000}{\pi r^2}$$

Then

$$S = 2\pi r^2 + 2\pi r \left( \frac{1000}{\pi r^2} \right) = 2\pi r^2 + \frac{2000}{r}$$
Thus the mathematical problem becomes:

Find the absolute minimum of the function \( S(r) = 2\pi r^2 + \frac{2000}{r} \) in the interval \((0, \infty)\).

\[
S'(r) = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}
\]

In the interval \((0, \infty)\), this function has only the critical point

\[
r_0 = 3\sqrt{\frac{500}{\pi}} \approx 5.419
\]

It is clear that the derivative of \( S \) is positive for \( r > r_0 \), and negative for \( r < r_0 \). Thus by the first derivative test, \( S \) has a relative minimum at \( r_0 \). Thus it has an absolute minimum there.
Here is the graph of the function $S$.

When $r = r_0$, we have $h_0 = \frac{1000}{\pi r_0^2}$. However, since $(r_0)^3 = \frac{500}{\pi}$, it follows that $r_0 = \frac{500}{\pi r_0^2} = \frac{h_0}{2}$. At the most efficient size, the height and the diameter of the base are equal, and are approximately 10.83 cm.
Example. A box shaped wire frame consists of two identical wire squares whose vertices are connected by four straight wires of equal length. If the frame is to be made from a wire of total length $L$, what should the dimensions be to obtain the box of greatest volume?

Solution. The volume is $V = x^2y$, and the total length of the wires is $L = 8x + 4y$. Thus we see that $y = \frac{L}{4} - 2x$
And therefore we need to maximize

\[ V = x^2 y = \frac{x^2 L}{4} - 2x^3 \]

Subject to the restriction that \( 0 \leq x \leq L/8 \).

\[ \frac{dV}{dx} = \frac{xL}{2} - 6x^2 = x \left( \frac{L}{2} - 6x \right) \]

This is 0 at \( x = 0 \) and \( x = L/12 \). If we consider the three points 0, \( L/12 \), \( L/8 \), we see that the respective values of \( V \) are:

\[
V(0) = 0 \quad V \left( \frac{L}{12} \right) = \frac{L^3}{1728} \quad V \left( \frac{L}{8} \right) = 0
\]

The maximum volume then occurs when \( x = L/12 \). In this case we also have \( 4y = L - 8L/12 = L/3 \), and so \( y = L/12 \). The optimal box is a cube whose volume is \( L^3/1728 \).
**Example.** Show that a right circular cylinder of greatest volume inscribed in a right circular cone has volume equal to $\frac{4}{9}$ of the volume of the cone.
By similar triangles, we have:

\[
\frac{h}{y} = \frac{r}{r-x}, \quad \text{so} \quad y = \frac{h}{r}(r-x)
\]

\[
V = V_{\text{cyl}} = \pi x^2 y = \pi x^2 \left( \frac{h}{r} \right) (r-x) = \pi h x^2 - \frac{\pi h x^3}{r}
\]

We must maximize this expression in the interval \([0, r]\).

\[
\frac{dV}{dx} = 2\pi h - \frac{3\pi h x^2}{r} = \pi h x \left( 2 - \frac{3x}{r} \right)
\]

This is 0 when \(x = 0\) or \(x = \frac{2r}{3}\).

The volume of the cylinder is 0 at both end points, 0 and \(r\). At \(2r/3\) we have
\[
y = \left( \frac{h}{r} \right) \left( r - \frac{2r}{3} \right) = \frac{h}{3}
\]

Then the volume of the cylinder is
\[
V = \pi \left( \frac{2r}{3} \right)^2 \frac{h}{3} = \frac{4\pi r^2 h}{27}
\]

The volume of the cone is
\[
V_{\text{cone}} = \frac{1}{3} \pi r^2 h
\]

It follows that the ratio \( \frac{V}{V_{\text{cone}}} \) is
\[
\left(\frac{\frac{4\pi r^2 h}{27}}{\frac{3}{\pi r^2 h}}\right) = \frac{4}{9}
\]
Example. Find all points on the curve $x^2 - y^2 = 1$ that are closest to the point (0, 2).
Let \((x, y)\) be any point on the curve. The distance between \((x, y)\) and \((0, 2)\) is given by the formula:

\[
\rho = \sqrt{(x-0)^2 + (y-2)^2} = \sqrt{x^2 + y^2 - 4y + 4}
\]

Rather than trying to minimize this function, it is clearly enough to find the places where \(\rho^2 = x^2 + y^2 - 4y + 4\) is a minimum.

The constraint equation is that \(x^2 - y^2 = 1\), so

\[
\rho^2 = (1 + y^2) + y^2 - 4y + 4 = 2y^2 - 4y + 5
\]

We must therefore find the absolute minimum of

\[
s(y) = 2y^2 - 4y + 5 \text{ on the interval } (-\infty, \infty), \text{ where } s \text{ is } \rho^2.
\]
\[
\frac{ds}{dy} = 4y - 4 = 4(y - 1)
\]

Hence the only critical point of the function is at \( y = 1 \), and this is a local minimum by the first derivative test. Therefore it is an absolute minimum. The closest points therefore have \( y = 1 \), and \( x = \pm \sqrt{2} \).

They are \((\sqrt{2},1)\) and \(( -\sqrt{2},1)\).

At these points, \( s = 3 \), so the actual minimum distance is \( \sqrt{3} \).