Length of a Plane Curve

Suppose that \( y = f(x) \) is a smooth curve on the interval \([a, b]\). Choose points \( x_1, \ldots, x_n \) and let \( P_k \) be the point above \( x_k \) on the curve. Suppose that the length of the segment connecting \( P_k \) and \( P_{k+1} \) is \( L_k \). Then the length of the curve is approximated by

\[
\sum_{k=1}^{n-1} L_k
\]
Clearly we have

\[
(\Delta x_k)^2 + (\Delta y_k)^2 = (L_k)^2
\]

So that

\[
L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}
\]

Thus the length of the curve is approximated by

\[
L \approx \sum_{k=1}^{n-1} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}
\]
Now suppose that the curve is smooth, that is $y = f(x)$ has a continuous derivative in $(a, b)$. Then between $x_k$ and $x_{k+1}$ there is a point $x_k^*$ with $f'(x_k^*) = \frac{\Delta y_k}{\Delta x_k}$.

We have:

\[
L \approx \sum_{k=1}^{n-1} \sqrt{\left(\Delta x_k\right)^2 + \left(\Delta y_k\right)^2} = \sum_{k=1}^{n-1} \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \Delta x_k = \sum_{k=1}^{n-1} \sqrt{1 + f'(x_k^*)^2} \Delta x_k
\]
This is clearly a Riemann sum for the integral
\[
\int_a^b \sqrt{1+\left[f'(x)\right]^2} \, dx = \int_a^b \sqrt{1+\left[\frac{dy}{dx}\right]^2} \, dx
\]

Thus:

**Theorem.** Suppose that \( y = f(x) \) is a smooth curve on the interval \([a, b]\) (meaning that the function is continuous on \([a, b]\) and its derivative exists and is continuous on \((a, b)\)). Then the length of the curve \( y = f(x) \) between \( a \) and \( b \) is given by
\[
\int_a^b \sqrt{1+\left[f'(x)\right]^2} \, dx = \int_a^b \sqrt{1+\left[\frac{dy}{dx}\right]^2} \, dx
\]
How to set up an arc length integral using infinitesimals.

If you zoom in on a smooth curve sufficiently far, it will appear to be a straight line segment. Thus if you start at any point $x$, then move an infinitesimal distance $dx$, the height of the curve increases by $dy$ and the following diagram summarizes the situation.

Thus $ds^2 = dx^2 + dy^2$
Manipulating formally, we have

\[ ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

Thus

\[ S = \int_a^b ds = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

As we saw before. The only thing that needs to be remembered is the equation

\[ ds^2 = dx^2 + dy^2 \]
Example. Find the length of the curve $y = x^2$ from $(0, 0)$ to $(4, 8)$.

Solution.

$$S = \int_{0}^{4} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_{0}^{4} \sqrt{1 + \left( \frac{3}{2} \cdot x^{-2} \right)^2} \, dx = \int_{0}^{4} \sqrt{1 + \frac{9}{4} x} \, dx$$

Let $u = 1 + \frac{9}{4} x$, $du = \frac{9}{4} \, dx$. Then

$$S = \frac{4}{9} \int_{1}^{10} \sqrt{u} \, u \, du = \frac{4}{9} \int_{3}^{10} \sqrt{u} \, du = \frac{4}{9} \left[ \frac{2}{3} u^{3/2} \right]_{3}^{10} = \frac{8}{27} \left[ 10^2 - 3^2 \right] = \frac{8}{27} \left[ \sqrt{1000} - 1 \right]$$
Example. Find the length of the curve \( y = \frac{x^6 + 8}{16x^2} \) from \( x = 2 \) to \( x = 3 \).

Solution. \[
\frac{d}{dx} \left[ \frac{x^6 + 8}{16x^2} \right] = \frac{dy}{dx} \left[ \frac{x^4}{16} + \frac{x^{-2}}{2} \right] = x^3 - x^{-3}
\]

\[
1 + \left( \frac{dy}{dx} \right)^2 = 1 + \frac{x^6}{16} - \frac{1}{2} + x^{-6} = \frac{x^6}{16} + \frac{1}{2} + x^{-6} = \left( \frac{x^3}{4} + x^{-3} \right)^2
\]

\[
S = \frac{3}{2} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \cdot dx = \left[ \frac{3x^3}{4} + x^{-3} \right] dx = \left[ \frac{x^4}{16} - \frac{x^{-2}}{2} \right]_2^3 = \frac{595}{144}
\]
In a similar way, we can find the length of a curve whose equation is $x = g(y)$. As before, we begin with

$$ds^2 = dx^2 + dy^2$$

Manipulating formally as before, we have

$$ds = \sqrt{dx^2 + dy^2} = \left[ \sqrt{\left(\frac{dx}{dy}\right)^2 + 1} \right] dy = \left[ \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \right] dy$$

Thus

$$S = \int_c^d ds = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$
Example. Find the length of the curve

$$x = \frac{1}{3} \left( y^2 + 2 \right)^{\frac{3}{2}}$$

from \( y = 0 \) to \( y = 1 \).

Solution.

$$\frac{dx}{dy} = \frac{1}{2} \left( y^2 + 2 \right)^{\frac{1}{2}} \left( 2y \right) = y \left( y^2 + 2 \right)^{\frac{1}{2}}$$

$$1 + \left( \frac{dx}{dy} \right)^2 = 1 + y^2 \left( y^2 + 2 \right) = y^4 + 2y^2 + 1 = \left( y^2 + 1 \right)^2$$

$$S = \int_0^1 \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = \int_0^1 \left[ y^2 + 1 \right] \, dy = \left[ \frac{y^3}{3} + y \right]_0^1 = \frac{4}{3}$$
We can also find the length of curves defined parametrically as $x = x(t), y = y(t)$. We start with the usual formula

$$ds^2 = dx^2 + dy^2$$

Then

$$ds = \sqrt{dx^2 + dy^2} = \left[ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \right] dt$$

Thus

$$S = \int_{t_1}^{t_2} ds = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$
Example. Find the length of the circumference of a circle of radius $r$ using the parametrization $x = r\cos(t), y = r\sin(t)$ from $t = 0$ to $t = 2\pi$.

Solution.

\[
\frac{dx}{dt} = -r\sin(t); \quad \frac{dy}{dt} = r\cos(t)
\]

\[
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2\cos^2(t) + r^2\sin^2(t) = r^2
\]

\[
S = \int_{0}^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \ dt = \int_{0}^{2\pi} r \ dt = 2\pi r
\]
Example. Find the length of the parametrized curve

\[ x = (1+t)^2, \quad y = (1+t)^3 \]

from \( t = 0 \) to \( t = 1 \).

Solution.

\[
\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 = \left[ 2(1+t) \right]^2 + \left[ 3(1+t)^2 \right]^2 = (1+t)^2 \left[ 4 + 9(1+t)^2 \right]
\]
\[ S = \int_0^1 \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = \int_0^1 (1+t)\sqrt{4+9(1+t)^2} \, dt \]

Let \( u = 4+9(1+t)^2; \) \( du = 18(1+t)dt \)

\[ S = \int_{13}^{40} \sqrt{u} \, du = \frac{1}{18} \cdot \frac{2}{3} \left[ u^{3/2} \right]_{13}^{40} = \frac{1}{27} \left[ 80\sqrt{10} - 13\sqrt{13} \right] \]