Surface area

When a curve is revolved around an axis, the curve itself generates a surface. This is the boundary of the corresponding volume of revolution.
Every infinitesimal element of arc $ds$ generates an infinitesimal element of surface $dS$.

The surface area of this infinitesimal area is $dS = 2\pi l ds$. 
Thus if the arc is part of a curve \( y = f(x) \), then \( l = y \) and we have

\[
dS = 2\pi y ds = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

so

\[
S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]
If the arc is part of the curve $x = g(y)$, then a similar analysis shows that

$$dS = 2\pi x ds = 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$
Example. Find the area of the surface generated by revolving the curve $y = \sqrt{x}$ from $x = 1$ to $x = 4$ about the $x$-axis.

Solution.

$$dS = 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2\pi \sqrt{x} \sqrt{1 + \frac{1}{4x}} \, dx = 2\pi \sqrt{x + \frac{1}{4}} \, dx$$

$$S = 2\pi \int_{1}^{4} \sqrt{x + \frac{1}{4}} \, dx = 2\pi \left[ \frac{17}{4} \sqrt{u} \sqrt{u} \frac{3}{u^2} \left[ \frac{17}{4} \right]^{5/4} \right]_{5/4}^{5/4} = \pi \left[ \frac{17\sqrt{17} - 5\sqrt{5}}{6} \right]$$
Example. Find the area of the surface of a sphere by regarding the sphere as generated by revolving a semicircle around an axis.

Solution. The sphere of radius $r$ can be generated by revolving around the $x$ axis the semicircle

$$y = \sqrt{r^2 - x^2}$$

This picture shows the case for $r = 2$. 
\[ \frac{dy}{dx} = \frac{-2x}{2\sqrt{r^2-x^2}} = \frac{-x}{\sqrt{r^2-x^2}} \]

\[ dS = 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = 2\pi \sqrt{r^2-x^2} \sqrt{1 + \frac{x^2}{r^2-x^2}} \, dx \]

\[ = 2\pi \sqrt{r^2-x^2} \sqrt{\frac{r^2}{r^2-x^2}} \, dx = 2\pi r \, dx \]

\[ S = 2\pi \int_{-r}^{r} r \, dx = 2\pi r x \bigg|_{-r}^{r} = 4\pi r^2 \]
Problem. Find the area of the surface generated by revolving around the y axis the curve \( x=2\sqrt{1-y} \) for \( y \) between \(-1\) and 0.

Solution.

\[
dS = 2\pi x \, ds = 4\pi \sqrt{1-y} \sqrt{1 + \left( \frac{-1}{\sqrt{1-y}} \right)^2} \, dy = 4\pi \sqrt{1-y} \sqrt{\frac{2-y}{1-y}} \, dy
\]

\[
= 4\pi \sqrt{2-y} \, dy
\]

\[
S = 4\pi \int_{-1}^{0} \sqrt{2-y} \, dy = 4\pi \int_{\frac{2}{3}}^{\frac{3}{2}} \sqrt{u} (-du)
\]

\[
= 4\pi \int_{\frac{3}{2}}^{\frac{2}{3}} \sqrt{u} (-du) = 4\pi \int_{\frac{3}{2}}^{\frac{2}{3}} \sqrt{u} \, du = \frac{8\pi (3\sqrt{3} - 2\sqrt{2})}{3}
\]
Example. Find the area of a cone of height $h$ and radius $r$ by regarding it to be generated by revolving the straight line shown below about the $x$ axis.
Solution. The cone can be generated by revolving the graph of the function

\[ y = \frac{r}{h} x \]

around the \( x \) axis.

\[
dS = 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = 2\pi \left( \frac{r}{h} x \right) \sqrt{1 + \frac{r^2}{h^2}} \, dx
\]

\[
S = 2\pi \frac{r}{h} \sqrt{1 + \frac{r^2}{h^2}} \int_0^h x \, dx = \left[ 2\pi \frac{r}{h} \sqrt{1 + \frac{r^2}{h^2}} \frac{x^2}{2} \right]_0^h = \left[ 2\pi \frac{r}{h} \sqrt{1 + \frac{r^2}{h^2}} \frac{h^2}{2} \right] = \pi r \sqrt{h^2 + r^2}
\]

\[
2\pi \frac{r}{h} \sqrt{1 + \frac{r^2}{h^2}} \frac{h^2}{2} = \frac{2\pi rh^2 \sqrt{h^2 + r^2}}{2h^2} = \pi r \sqrt{h^2 + r^2}
\]
You can also find surface area generated by a parametrically defined curve, in some cases.

**Example.** Find the area of the surface of a sphere by regarding the sphere as generated by revolving a semicircle around an axis. Parametrize the semicircle.

We take the parametric form to be: \[
\begin{cases}
  x = r \cos(\theta), \\
  y = r \sin(\theta),
\end{cases}
\] for \(0 \leq \theta \leq \pi\).
Then the surface area is:

\[ dS = 2\pi \ yds = 2\pi \ r \sin(\theta) \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \ d\theta \]

\[ = 2\pi \ r \sin(\theta) \sqrt{r \cos(\theta)^2 + r \sin(\theta)^2} \ d\theta = 2\pi \ r^2 \sin(\theta) \ d\theta \]

\[ S = 2\pi \ r^2 \int_{0}^{\pi} \sin(\theta) \ d\theta = 2\pi \ r^2 \left[ \cos(0) - \cos(\pi) \right] = 4\pi \ r^2 \]