**Work**

**Definition.** If a constant force $F$ is exerted on an object, and as a result the object moves a distance $d$ in the direction of the force, then the work done is $Fd$.

**Example.** If a block is pushed by a constant force of 200 lb. Through a distance of 20 ft., then the total work done is 4000 ft-lbs.
If the work varies with the distance, we can use an infinitesimal approach.

At any point \( x \), the force is \( F(x) \). This force can be regarded as constant as we move from \( x \) to \( x + dx \), a distance of \( dx \) units. The work done in moving this infinitesimal distance is therefore \( dW = F(x)dx \), and we can find the total work done over the entire distance, as usual, by adding up these infinitesimal pieces of work with an integral. This leads to the following theorem.

**Theorem.** If a varying force \( F(x) \) is applied to an object moving in a straight line in the direction of the force, over an interval \([a, b]\), the total work done is

\[
W = \int_{a}^{b} F(x)dx
\]
Problem. The figure below is the graph of a variable force $F(x)$ in the positive $x$-direction. Find the work done by the force on a particle that moves from $x = 0$ to $x = 5$. 

![Graph of force vs. position](image)
**Solution.** Over the first 2 meters, the force is constant at 40 N. The work is therefore 80 J. Over the next three meters the force has formula

\[
F(x) = -\frac{40}{3}(x-2) + 40 = -\frac{40}{3}x + \frac{200}{3}
\]

The work done by this force in moving a distance \(dx\) from \(x\) is \(dW = F(x)dx\). Thus the total work done in this part of the movement is

\[
\int_2^5 \left(-\frac{40}{3}x + \frac{200}{3}\right) dx = \left[-\frac{40}{3}x^2 + \frac{200}{3}x\right]_2^5 = \frac{1000}{6} - \frac{640}{6} = 60 \text{ J.}
\]

The total work is therefore 140 J.
Sometimes the force applied to an object varies with the location of the object. For example, Hooke’s Law states that the force necessary to hold a stretched spring $x$ units beyond its natural length is $kx$, where $k$ is a constant called the *spring constant*. 
Problem. How much work is done when stretching a spring of spring constant $k$ an amount $x_0$ from its natural length $L$?

Solution. Arrange the coordinate system so that the right hand end of the spring is at $x = 0$. Then the spring is expanded so that its end is at $x_0$. At each point $x$ between 0 and $x_0$ the Force being applied is $kx$. When we then stretch the spring an infinitesimal amount $dx$, we do an infinitesimal amount of work $dW = kxdx$ (since the force can be regarded as constant during the motion from $x$ to $x + dx$).

The total amount of work done is therefore

$$k \int_{0}^{x_0} x \, dx = \left. \frac{kx^2}{2} \right|_{0}^{x_0} = \frac{kx_0^2}{2}.$$
Problem. A spring whose natural length is 15 cm exerts a force of 45 N when stretched to a length of 20 cm.

(a) Find the spring constant (in newtons / meter).

(b) Find the work done in stretching the spring 3 cm beyond its natural length.

(c) Find the work done in stretching the spring from a length of 20 cm to a length of 25 cm.

Solution. (a) The change of length is 5 cm = .05 m. Since $F=kx$ we have $45 = k(.05)$, so $k = 4500/5 = 900$ newtons/meter.

(b) The work done in moving a distance $d$ is $\frac{kd^2}{2}$.

Thus the work done in stretching 3 cm is $\frac{900(.03)^2}{2} = .405J$. 
(c) The work done is

\[ 900 \int_{0.05}^{0.1} x \, dx = \left. \frac{900x^2}{2} \right|_{0.05}^{0.1} = 450 \left[ (0.1)^2 - (0.05)^2 \right] = 3.375 \text{J} \]
How much work is done when a cylindrical tank of radius $r$ and height $h$ filled with a fluid of density $\rho$ lbs. per cubic ft. is pumped empty through a pipe at the top?
We imagine the water divided into infinitesimally thick disks of volume $\pi r^2 dx$. If such a slice is at a height of $x$ ft. from the base, it can only be moved to the top with a force equal to its wt., which is $dF = \pi r^2 \rho dx$. The total work done in that process is therefore

$$dW = \pi r^2 \rho (h-x) dx.$$ 

The work done in this process is therefore
\[ W = \int_0^h dW = \pi r^2 \rho \int_0^h (h-x) \, dx = \pi r^2 \rho \left[ h x - \frac{x^2}{2} \right]_0^h = \pi r^2 \rho \frac{h^2}{2} \]
Problem. A cylindrical tank of radius 5 ft and height 9 ft is two-thirds filled with water. Find the work required to pump all the water over the upper rim.

Solution. A small disk of water of thickness $dx$ at height $x$ ft from the bottom weighs $dF = \pi (25)(62.4)dx$ lbs. This force must be applied over a distance $(9 - x)$ to move this water over the rim. The resulting work in ft.-lbs. is $dW = \pi (25)(62.4)(9-x)dx$.

Thus the total work done in emptying the tank is

$$W = \pi (25)(62.4) \int_{0}^{6} (9-x)dx = (25)(62.4)\pi \left[ 9x - \frac{x^2}{2} \right]_0^6 = 1560\pi [36] = 56160\pi$$

ft-lbs.
Problem. A cone shaped reservoir is 20 ft in diameter at the top and 15 ft deep. If the reservoir is filled to the depth of 10 ft, how much work is required to pump all the water to the top?

Solution.
Clearly, by similar triangles, \( \frac{r}{y} = \frac{10}{15} = \frac{2}{3} \).

Then \( r = \frac{2}{3} y \)

The volume of the infinitesimal disk of water shown is

\[ dV = \pi r^2 dy = \frac{4\pi}{9} y^2 dy, \]

and so the weight (and the force needed to lift the disk) is

\[ dF = (62.4) \frac{4\pi}{9} y^2 dy. \]
The height the disk must be lifted is 15 – y ft. The work done in this process is then

\[ dw = (62.4)\frac{4\pi}{9} (15 - y) y^2 dy. \]

The total work is therefore.

\[
W = (62.4)\frac{4\pi}{9} \int_0^{10} (15y^2 - y^3) dy = \frac{249.6\pi}{9} \left[ 5y^3 - \frac{y^4}{4} \right]_0^{10} = \frac{208000\pi}{3} \text{ ft-lbs.}
\]
**Problem.** Coulomb’s law implies that like electrostatic charges repel one another with a force inversely proportional to the square of the distance between them. Suppose that two charges $A$ and $B$ repel with a force of $k$ newtons when they are positioned at points $A(–a, 0)$ and $B(a, 0)$, where $a$ is measured in meters.

Find the work $W$ required to move charge $A$ along the $x$–axis to the origin if charge $B$ remains stationary.
Solution.

The force between the particles at distance $d$ is $\frac{c}{d^2}$, where $c$ is a constant. Since the initial force is $k$, we see that $k = \frac{c}{4a^2}$, so $c = 4ka^2$.

Thus the force between the particles at distance $d$ is

$$\frac{4k a^2}{d^2}$$
Suppose that the particle at $A$ has been moved to $x$. When it is moved an additional amount $dx$, the force over that infinitesimal distance can be taken to be

$$dF = \frac{4ka^2}{(a-x)^2}$$

Thus the work done in moving the particle from $x$ to $x + dx$ is

$$dW = \frac{4ka^2}{(a-x)^2} \, dx$$
The total work done is therefore

\[ W = \int_{-a}^{0} \frac{4ka^2}{(a-x)^2} dx = \left( \frac{4ka^2}{a-x} \right)_{-a}^{0} = \left[ 4ka - 2ka \right] = 2ka \ \text{J}. \]