Integration by Substitution

The method of substitution can often be used to simplify complicated integration problems. It is based on the chain rule for differentiation.

Suppose that $F$ is an antiderivative of $f$, and $g$ is a differentiable function. Then

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

by the chain rule.

In integral form, we can write this as

$$\int F'(g(x))g'(x)dx = F(g(x)) + C$$

or

$$\int f(g(x))g'(x)dx = F(g(x)) + C \quad (1)$$
To see how to use this we make a substitution, letting \( u = g(x) \). Then we see that \( du = g'(x)dx \). Looking at formula (1) we see

\[
\int f(g(x))g'(x)dx = F(g(x)) + C \quad \text{(1)}
\]

So, formula (1) becomes

\[
\int f(u)du = F(u) + C \quad \text{(2)}
\]

**Definition:** The process of turning (1) into (2) by making a substitution \( u = g(x) \), \( du = g'(x)dx \) is called the *method of u-substitution.*
Example. Evaluate the indefinite integral
\[ \int (x^2 + 1)^{50} (2x) \, dx \]

Solution. We know how to handle powers of a variable, so the above formula suggests that we make the substitution

\[ u = (x^2 + 1) \quad \text{and} \quad du = 2x \, dx \]

Then

\[ \int (x^2 + 1)^{50} (2x) \, dx = \int u^{50} \, du = \frac{u^{51}}{51} + C = \frac{(x^2 + 1)^{51}}{51} + C \]

Here we had a fortunate situation where the division of the integral into \( du \) plus parts expressible in terms of \( u \) was immediate. Often some manipulation must take place.
**Example.** Evaluate the indefinite integral
\[ \int (x^2 + 1)^{30} \, dx \]

**Solution.** Here we feel that we should make a similar substitution to the one made previously. We let
\[ u = (x^2 + 1) \quad du = 2x \, dx \]
Then the integrand becomes \( u^{30} \), but we do not have \( du \) remaining. We need to express the rest of the integral in terms of \( u \) and \( du \). Since \( du = 2x \, dx \) we see that \( x \, dx = \frac{1}{2} du \). Then

\[
\int (x^2 + 1)^{30} \, dx = \int u^{30} \frac{1}{2} \, du = \frac{1}{2} \int u^{30} \, du = \frac{1}{2} \frac{u^{31}}{31} + C = \frac{(x^2 + 1)^{31}}{62} + C
\]
In the method of $u$-substitution, you have control over the choice of $u$, but once that choice is made it must be possible to express all parts of the original integral, including the differential, in terms of $u$ and $du$. There can be no extra $x$ part remaining.

Extra terms involving $x$ that might be left over cannot be brought outside the integral sign - they must be expressed in terms of $u$. 
Example. Try to evaluate the indefinite integral
\[ \int (x^3 + 1)^{30} \, x \, dx \]
by the method of substitution.

Solution. As before, we attempt the substitution
\[ u = (x^3 + 1) \quad \text{du} = 3x^2 \, dx \]

Then the integrand becomes \( u^{30} \), but there is no way to express the remaining \( x \, dx \) in terms of \( du \). Some students would attempt to say that
\[ du = 3x^2 \, dx = 3x(\, dx) \]

And therefore that
\[ x \, dx = \frac{du}{3x} \]
Then the integral would become

\[ \int (x^3 + 1)^{30} x \, dx = \int u^{30} \frac{du}{3x} \]

This is hopeless since you cannot move the x term outside the integral sign.

In fact, the integral cannot be done at all by this method.
**Example.** What would work is to evaluate the indefinite integral

\[ \int (x^3 + 1)^{30} x^2 \, dx \]

by the method of substitution.

**Solution.** We attempt the substitution

\[ u = (x^3 + 1) \quad du = 3x^2 \, dx \]

This time the integrand becomes \( u^{30} \), and since

\[ du = 3x^2 \, dx \quad \text{we have} \quad x^2 \, dx = \frac{1}{3} du \]

Therefore the integral becomes:

\[ \int (x^3 + 1)^{30} x^2 \, dx = \int u^{30} \frac{1}{3} \, du = \frac{1}{3} \int u^{30} \, du = \frac{u^{31}}{93} + C = \frac{(x^3 + 1)^{31}}{93} + C \]
We summarize the steps in the method.

**Integration by substitution.**

**Step1.** Make a choice for $u$, say $u = g(x)$

**Step2.** Compute $du/dx = g'(x)$

**Step3.** Make the substitution $u = g(x)$, $du = g'(x)dx$

At this point the entire integral must be in terms of $u$; no $x$’s should remain. If they do, try a different substitution.

**Step4.** Evaluate the resulting integral if possible.

**Step5.** Replace $u$ by $g(x)$, so that the final answer is in terms of $x$. 
Example. Use the method of substitution to evaluate the indefinite integral
\[ \int \sin(5x) \, dx \]

Solution. To reduce the function to a familiar one, we make the substitution
\[ u = 5x \quad du = 5 \, dx \]

Then we have \( dx = \frac{1}{5} \, du \) so the integral becomes:

\[ \int \sin(5x) \, dx = \int \sin(u) \frac{1}{5} \, du \]

\[ = \frac{1}{5} \int \sin(u) \, du = -\frac{1}{5} \cos(u) + C = -\frac{1}{5} \cos(5x) + C \]
Example. Use the method of substitution to evaluate the indefinite integral
\[ \int \sin^2(x) \cos(x) \, dx \]

Solution. One must always ask the question - “If I choose this value for \( u \), can I find \( du \) in the integral expression?”

We note that the differential of \( \sin(x) \) is \( \cos(x) \, dx \), which is present in the above integral. This suggests that we make the substitution
\[ u = \sin(x) \quad \Rightarrow \quad du = \cos(x) \, dx \]

Then the integral becomes:
\[ \int \sin^2(x) \cos(x) \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sin^3(x)}{3} + C \]
Example. Use the method of substitution to evaluate the indefinite integral

\[ \int \sec^2(4x+1) \, dx \]

Solution. A useful principle is “Get rid of anything causing trouble by substituting for it”

In this integral, we would be fine if we had a variable in the \( \sec^2 \) term, rather than a more complicated expression. Thus we choose the substitution

\[ u = 4x + 1 \quad \text{and} \quad du = 4 \, dx \]

Then \( dx = \frac{1}{4} \, du \) and the integral becomes:

\[ \int \sec^2(4x+1) \, dx = \int \sec^2(u) \frac{1}{4} \, du = \frac{1}{4} \int \sec^2(u) \, du \]

\[ = \frac{1}{4} \tan(u) + C = \frac{1}{4} \tan(4x+1) + C \]
**Example.** Use the method of substitution to evaluate the indefinite integral

\[ \int (2x+7)(x^2+7x+3)^5 \, dx \]

**Solution.** Some quantity is being raised to a power. If we substitute for it, we can use the power rule for integration - provided that we can find its differential. We try the substitution:

\[ u = x^2 + 7x + 3 \quad du = (2x+7)\,dx \]

The integral becomes:

\[ \int (2x+7)(x^2+7x+3)^5 \, dx = \int u^5 \, du = \frac{4}{5}u^5 + C = \frac{4}{5}(x^2+7x+3)^5 + C \]
**Example.** Use the method of substitution to evaluate the indefinite integral

\[ \int \frac{e^x}{1+e^x} \, dx \]

**Solution.** We might first consider the substitution \( u = e^x \) \( du = e^x \, dx \).

The integral becomes:

\[ \int \frac{e^x}{1+e^x} \, dx = \int \frac{1}{1+u} \, du \]

This integral could be evaluated by making a further substitution for \( u + 1 \). However, it is easier to just begin instead with the substitution: \( u = 1 + e^x \) \( du = e^x \, dx \).

Then the integral becomes

\[ \int \frac{e^x}{1+e^x} \, dx = \int \frac{1}{u} \, du = \ln(|u|) + C = \ln\left(|1+e^x|\right) + C \]

**Note.** The absolute value could be omitted here, since the expression \( 1 + e^x \) is always positive.
Integration Hint

One thing to look for in an integrand that is a fraction is whether the numerator is equal to the derivative of the denominator (or is a multiple of it). In that case, substitute for the denominator.

The resulting integral will be of the form

$$\int \frac{1}{u} \, du$$

And this integrates to the natural logarithm of the absolute value of $u$. 
Example. Use the method of substitution to evaluate the indefinite integral

\[ \int x^2 \sqrt{1+x} \, dx \]

Solution. The problems in this integral seem to stem from the fact that the square root has a more complicated expression than \( x \) inside it. Thus we will get rid of this expression by substitution. Let \( u = 1 + x \), \( du = dx \)

Then the integral becomes:

\[
\int x^2 \sqrt{1+x} \, dx = \int (u-1)^2 \sqrt{u} \, du = \int (u^2 - 2u + 1) \sqrt{u} \, du = \int u^2 \sqrt{u} - 2u \sqrt{u} + \sqrt{u} \, du
\]

\[
= \frac{5}{7} u^2 - 2 \frac{3}{2} u^2 + 2 \frac{1}{3} u^2 + C
\]

\[
= \frac{2}{7} u^2 - 4 \frac{5}{3} u^2 + 2 \frac{3}{5} u^2 + C = \frac{2}{7} (1+x)^2 - 4 \frac{5}{3} (1+x)^2 + 2 \frac{3}{5} (1+x)^2 + C
\]
**Example.** Use the method of substitution to evaluate the indefinite integral
\[ \int e^{\tan x} \sec^2 x \, dx \]

**Solution.** The main problem in this integral is the presence of the expression \( \tan x \) in the exponential. Since the differential of this expression is present we make the substitution.

\[ u = \tan x \quad du = \sec^2 x \, dx \]

Then the integral becomes:

\[ \int e^{\tan x} \sec^2 x \, dx = \int e^u \, du = e^u + C = e^{\tan x} + C \]
Example. Use the method of substitution to evaluate the indefinite integral
\[ \int \frac{x}{\sqrt{4-5x^2}} \, dx \]

Solution. The main problem in this integral appears to be the presence of the complicated expression under the square root. Let us try the substitution.

\[ u = 4 - 5x^2 \quad du = -10x \, dx \]

The expression \( x \, dx \) is then equal to \( -\frac{1}{10} \, du \) so we have

\[ \int \frac{x}{\sqrt{4-5x^2}} \, dx = \int \frac{1}{\sqrt{u}} \left( -\frac{1}{10} \right) du = -\frac{1}{10} \int u^{-\frac{1}{2}} du = -\frac{1}{10} \left( 2u^{\frac{1}{2}} \right) \]

\[ = -\frac{1}{5} \sqrt{u} + C = -\frac{1}{5} \sqrt{4-5x^2} + C \]
**Example.** Use the method of substitution to evaluate the indefinite integral

\[ \int x \cos(3x^2) \, dx \]

**Solution.** Let us substitute to get rid of the expression inside the cosine. Let 

\[ u = 3x^2 \quad du = 6x \, dx \]

The expression \( x \, dx \) is then equal to \( \frac{1}{6} \, du \) so we have

\[ \int x \cos(3x^2) \, dx = \int \cos(u) \frac{1}{6} \, du = \frac{1}{6} \int \cos(u) \, du \]

\[ = \frac{1}{6} \sin(u) + C = \frac{1}{6} \sin(3x^2) + C \]
Note that in many ways, we were lucky in the previous two examples to have the differential of \( u \) present.

If those two integrals had been the apparently simpler integrals

\[
\int \frac{dx}{\sqrt{4-5x^2}} \quad \text{and} \quad \int \cos(3x^2)dx
\]

then the first would require a more sophisticated substitution called a *trigonometric substitution* - we will learn this later - and the second integration cannot be done at all, at least in terms of elementary functions (functions whose names you know from calculus I).
Similarly, we cannot express the integral \( \int e^{-x^2} \, dx \) in terms of known functions. This integral is related by the fundamental theorem of calculus to the area under the curve

\[ y = e^{-x^2} \]

This curve plays a fundamental role in statistics. The indefinite integral can be evaluated numerically, and it is so important that it is given a name - the “error function” - and is written

\[ \text{erf}(x) \]
Example. Use the method of substitution to evaluate the indefinite integral 
\[ \int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx \]

Solution. Let us substitute to get rid of the square root inside the trigonometric expression. Let

\[ u = \sqrt{x} = x^{\frac{1}{2}} \quad du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{dx}{2\sqrt{x}} \]

The expression \( \frac{dx}{\sqrt{x}} \) is then equal to \( 2du \), so we have

\[ \int \frac{\sec^2 \sqrt{x}}{\sqrt{x}} \, dx = \int \sec^2(u)2du = 2\int \sec^2(u) \, du = 2\tan(u) + C = 2\tan(\sqrt{x}) + C \]