Area as a Limit

Let $R$ be the region under the graph of a positive function $f(x)$, between the lines $x = a$ and $x = b$. 
Procedure for defining and calculating the area of $R$

(a) Divide the interval $[a, b]$ into $n$ equal subintervals.

(b) Over each subinterval construct a rectangle whose height is the value of $f$ at any point in the subinterval.

(c) The area $A(R_n)$ of the rectangular complex $R_n$ formed by these rectangles can be regarded as an approximation to the “area” of the region $R$.

(d) Repeat this process using more and more subdivisions (larger and larger $n$).

(e) Define the area of $R$ to be the limit of the areas of the $R_n$ that is

$$A(R) = \lim_{{n \to \infty}} A(R_n)$$
We need to make this mathematically precise.

We divide the base into $n$ equal subintervals by placing $n - 1$ equally spaced points $x_1, x_2, \ldots, x_{n-1}$ in between $a$ and $b$. Each subinterval has length

$$\Delta x = \frac{b - a}{n}$$
We then choose points $x_1^*, x_2^*, \ldots, x_n^*$ within the subintervals. Draw rectangles over each subinterval, using the height of the graph at $x_i^*$ for the height of the $i$th rectangle.
The area of the rectangular complex $R_n$ is the sum of the areas of the rectangles, that is

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

or, using sigma notation,

$$A(R) = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

**Definition.** *(Area under a curve)*. If the function $f$ is continuous on $[a, b]$ and if $f(x) \geq 0$ for all $x$ in $[a, b]$, then the area under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A(R) = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$
Remarks.

1. To be considered an exact mathematical definition, we would have to prove that the limit mentioned in the definition of area exists, and its value does not depend on the choice of points $x_i^*$ chosen in the various subrectangles. This is done in more advanced courses.

2. This definition does not lead to an easy way to compute the area exactly except in very special cases. Having used the definition to define area precisely, we will find other ways to calculate areas.

3. In many cases, the definition given (or some refinement of it) remains the only way to calculate the area approximately (when an exact calculation is not possible or is too difficult). Then we approximate the area by choosing a large $n$ and using a computer.
**Example.** Use the definition to calculate precisely the area under the curve $y = x^2$ from $x = 1$ to $x = 2$.

Let us approximate by using the heights at the **left endpoints** of the subintervals to define the rectangles.

In the case where $n = 4$, we show the result on the left. The three added points are $\frac{5}{4}, \frac{6}{4},$ and $\frac{7}{4}$, and $\Delta x = \frac{2 - 1}{4} = \frac{1}{4}$. The approximate area is then

\[
\frac{1}{4} f(1) + \frac{1}{4} f\left(\frac{5}{4}\right) + \frac{1}{4} f\left(\frac{6}{4}\right) + \frac{1}{4} f\left(\frac{7}{4}\right)
\]
\[ \frac{1}{4} f(1) + \frac{1}{4} f\left(\frac{5}{4}\right) + \frac{1}{4} f\left(\frac{6}{4}\right) + \frac{1}{4} f\left(\frac{7}{4}\right) = \sum_{k=0}^{3} \frac{1}{4} f\left(1 + \frac{k}{4}\right) = \frac{1}{4} \sum_{k=0}^{3} \left(1 + \frac{k}{4}\right)^2 \]

In general, if we divide \([1, 2]\) into \(n\) intervals, each will have length

\[ \Delta x = \frac{1}{n} \]

If we again use the left end points of the intervals to find the heights of the rectangles, we see that these left hand points are.

\[ 1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, 1 + \frac{3}{n}, \ldots, 1 + \frac{n-1}{n} \]

Thus the approximate area is

\[ \sum_{k=0}^{n-1} \frac{1}{n} f\left(1 + \frac{k}{n}\right) = \frac{1}{n} \sum_{k=0}^{n-1} \left(1 + \frac{k}{n}\right)^2 = \frac{1}{n} \sum_{k=0}^{n-1} \left[1 + \frac{2k}{n} + \frac{k^2}{n^2}\right] \]
\[
\begin{align*}
&= \left[ \frac{1}{n} \sum_{k=0}^{n-1} 1 \right] + \left[ \frac{2}{n^2} \sum_{k=0}^{n-1} k \right] + \left[ \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 \right] \\
&= \left[ \frac{n}{n} \right] + \left[ \frac{2}{n^2} \cdot \frac{(n-1)n}{2} \right] + \left[ \frac{1}{n^3} \cdot \frac{(n-1)n(2n-1)}{6} \right] \\
&= 1 + \left[ \frac{n-1}{n} \right] + \left[ \frac{(n-1)(2n-1)}{6n^2} \right] \\
&= 1 + \left[ \frac{n-1}{n} \right] + \frac{1}{6} \left[ \frac{n-1}{n} \right] \left[ \frac{2n-1}{n} \right]
\end{align*}
\]

As \( n \) tends to infinity, this approximation tends to \( \frac{7}{3} \). Thus the area under the curve is \( \frac{7}{3} \).
In the previous example, we used the left hand end points of the subintervals to form the rectangles. Often the right hand end points or midpoints of the intervals are used instead.
Signed Area Under a continuous function.

In the definition of area given previously, we assumed that the continuous function $f$ had only nonnegative values, that is the area was entirely above the $x$-axis.

If $f$ can be both positive and negative on the interval $[a, b]$, then the limit we computed,

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$$

formed by dividing the interval into subintervals and choosing points in those subintervals to evaluate $f$, is not area.
Examples show that this limit tends to a number that represents the sum of all areas under the curve and above the axis minus the sum of those areas under the curve and below the axis. This is called the \textit{signed area under $f$ from $a$ to $b$}.
The limit \( \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x \) is so important that some notation is associated with it. We denote it by the symbol
\[
\int_{a}^{b} f(x) \, dx
\]
That is,
\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^*) \Delta x
\]
This number is called the \textit{definite integral of \( f \) from \( a \) to \( b \)}. If \( f \) is positive, this number is the area under the graph of \( f \) between \( a \) and \( b \). If not then it is the signed area under \( f \) from \( a \) to \( b \). In applications it represent many other important quantities.
Anatomy of definite integral or Riemann Integral

\[
\int_{a}^{b} f(x)\,dx
\]

- **Integral Sign**
- **Integrand** $f(x)$
- **Upper limit of integration** $b$
- **Lower limit of integration** $a$
**Problem.** Sketch the region whose signed area is represented by the definite integral
\[ \int_{0}^{2} \left( 1 - \frac{1}{2} x \right) dx \]
and evaluate the integral geometrically.

**Solution.**

\[ \int_{0}^{2} \left( 1 - \frac{1}{2} x \right) dx = \frac{1}{2} (2 \times 1) = 1 \]
**Problem.** Sketch the region whose signed area is represented by the definite integral
\[ \int_{-1}^{4} \left(1-\frac{1}{2}x\right)dx \]
and evaluate the integral geometrically.

**Solution.**

\[ \int_{-1}^{4} \left(1-\frac{1}{2}x\right)dx = A_1 - A_2 = \frac{1}{2} \left(3 \times \frac{3}{2}\right) - \frac{1}{2} (2 \times 1) = \frac{9}{4} - 1 = \frac{5}{4} \]