Rectilinear Motion; Average Value

Let $s(t)$ be the position of the particle on the line at time $t$, and $s_0$ be the initial position of the particle. At any time, the instantaneous velocity and acceleration of the particle are given by

$$v(t) = s'(t) = \frac{ds}{dt} \quad \text{and} \quad a(t) = v'(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

These formulas also mean that

$$s(t) = \int v(t) \, dt \quad \text{and} \quad v(t) = \int a(t) \, dt$$
Example. The position of a body moving along a straight line with time is given by \( s(t) = \sin(2t) \). Find expressions for the velocity and acceleration at any time.

Solution.

\[
\begin{align*}
v(t) &= s'(t) = \sin(2t)' = 2\cos(2t) \\
a(t) &= v'(t) = (2\cos(2t))' = -4\sin(2t)
\end{align*}
\]
Example. A body moves along a straight line in such a way that the acceleration at any time is \( a(t) = 2t \). If the body starts with initial position \( s(0) = 1 \) and initial velocity \( v(0) = 3 \), find expressions for the velocity and position at any time.

Solution.

\[
v(t) = \int a(t) \, dt = \int 2t \, dt = t^2 + C_1. \quad \text{Since } 3 = v(0) = C_1, \text{ we have}\]

\[
v(t) = t^2 + 3.
\]

\[
s(t) = \int v(t) \, dt = \int \left( t^2 + 3 \right) \, dt = \frac{t^3}{3} + 3t + C_2. \quad \text{Since } 1 = s(0) = C_2, \text{ we have}\]

\[
s(t) = \int v(t) \, dt = \int \left( t^2 + 3 \right) \, dt = \frac{t^3}{3} + 3t + 1.
\]
**Example.** A particle moves along the s axis. Use the given information to find the position function of the particle.

(a) \( v(t) = t^3; s(8) = 0 \)  
(b) \( a(t) = \sqrt{t}; v(4) = 1; s(4) = -5 \)

**Solution.**

(a) \( s(t) = \int v(t) \, dt = \int t^3 \, dt = \frac{3}{5} t^5 + C_1 \).

Since \( 0 = s(8) = \frac{96}{5} + C_1 \), \( C_1 = -\frac{96}{5} \) and \( s(t) = \frac{3}{5} t^5 - \frac{96}{5} \).
(b) \[ v(t) = \int a(t) \, dt = \int \sqrt{t} \, dt = \frac{2}{3} t^{3/2} + C. \]

Since \[ 1 = v(4) = \frac{16}{3} + C, \quad C = 1 - \frac{16}{3} = -\frac{13}{3} \]
and \[ v(t) = \frac{2}{3} t^{3/2} - \frac{13}{3}. \]

Then \[ s(t) = \int v(t) \, dt = \left[ \frac{2}{3} t^{3/2} - \frac{13}{3} \right] \, dt = \frac{4}{15} t^{5/2} - \frac{13}{3} t + C_2. \]

Since \[ -5 = s(4) = \frac{128}{15} - \frac{52}{3} + C_2 = -\frac{44}{5} + C_2, \quad C_2 = \frac{44}{5} - 5 = \frac{19}{5}, \]

Finally, \[ s(t) = \frac{4}{15} t^{5/2} - \frac{13}{3} t + \frac{19}{5}. \]
Uniformly Accelerated Motion

A very important case is that of constant acceleration.

**Theorem.** If a particle moves with a constant acceleration $a$ along an $s$-axis, and if the position and velocity at time $t = 0$ are respectively $s_0$ and $v_0$, then the position and velocity functions of the particle are

$$s(t) = s_0 + v_0 t + \frac{1}{2} at^2 \quad \text{and} \quad v(t) = v_0 + at$$

**Proof.**

$$v(t) = \int a \, dt = at + C_1.$$ Since $v_0 = v(0) = C_1$, we have $v(t) = v_0 + at$.

$$s(t) = \int \left( v_0 + at \right) \, dt + C_2 = v_0 t + \frac{at^2}{2} + C_2.$$ Since $s_0 = s(0) = C_2$

$$s(t) = s_0 + v_0 t + \frac{1}{2} at^2$$
**Problem:** A car traveling at 60 mph along a straight road decelerates at a constant rate of 10 ft/s².  
(a) How long will it take until the speed is 45 mph?  
(b) How far will the car travel before coming to a stop?  

**Solution.** (a) First of all we need to convert mi/hr to ft/s.  

\[
\begin{align*}
\frac{1\text{mi}}{\text{hr}} \times \frac{1\text{hr}}{3600\text{s}} \times \frac{5280\text{ft}}{\text{mi}} &= \frac{22\text{ft}}{15\text{s}} \\
\end{align*}
\]

This means that 60mph = 88ft/s and 45 mph = 66 ft/s.  

The constant acceleration is \(a = -10\), and the initial velocity is 88 ft/s. Thus \(v(0) = v_0 + at = 88 - 10t\).  

We see that if we set \(v = 45 \text{ mph} = 66 \text{ ft/s}\), and solve for \(t\), we will have \(t = \frac{88 - 66}{10} = 2.2\text{s}\).
(b) Since $v(t)=88-10t$ it is clear that the car will stop in 8.8s.

Take the initial position to be 0, so that $s(t)=s_0 + v_0 t + \frac{1}{2}at^2 = 88t - 5t^2$.

Then at $t = 8.8$ we have $s(8.8) = (88)(8.8) - 5(8.8)^2 = 387.2$ ft.
**Problem.** In the final sprint of a rowing race the challenger is rowing at a constant speed of 12 m/s. At the point where the leader is 100 m from the finish line and the challenger is 15 m behind, the leader is rowing at 8 m/s but starts accelerating at a constant 0.5 m/s². Who wins?

**Solution.** We start time at the moment that the leader begins to accelerate. At this moment the position of the leader is at $s = -100$m, the velocity is 8 m/s and the acceleration is 0.5 m/s². The position of the leader at subsequent time $t$ is then

$$s(t) = s_0 + v_0 t + \frac{1}{2} a t^2 = -100 + 8t + \frac{1}{4} t^2$$
To find the time it takes the leader to cross the finish line \((s = 0)\), we set \(s\) to 0 and solve the resulting quadratic equation.

\[-100+8t+\frac{1}{4}t^2=0\]  or  \(t^2+32t-400=0\). The solutions are \(t=-41.62\) and \(t=9.6125\) so we take the second.

In 9.6125 seconds the challenger travels \((9.6125)(12) = 115.35\) m. Since the challenger begins 115 m from the finish line, the challenger wins, since the challenger is already about a foot beyond the finish when the leader crosses.
The free fall model

The force of gravity near the surface of the earth is taken to be a constant \(-g\), where \(g = 9.8 \text{ m/s}^2\) or \(32 \text{ ft/s}^2\). Thus (neglecting friction) if a particle has an initial position and velocity of \(s_0\) and \(v_0\), resp, and moves subject only to the force of gravity, then the formula for its velocity at any time is

\[
v(t) = v_0 - gt
\]

and the formula for its position at any time \(t\) is

\[
s(t) = s_0 + v_0 t - \frac{gt^2}{2}
\]
\[ a = -g \]
**Problem.** A ball is thrown vertically upward from ground level with an initial velocity of 16 ft/s.
(a) How high will the ball rise?
(b) How long will it take for the ball to hit the ground?
(c) How long will the ball be moving upward?

**Solution.** Here we have $s_0 = 0$, $v_0 = 16$. Thus

$$v(t) = 16 - 32t \quad \quad s(t) = 16t - 16t^2$$

We first solve (c). The ball rises until it stops ($v = 0$), which occurs at $t = 1/2$ s.
When $t = 1/2$, we see that $s(.5) = 16(.5) - 16(.25) = 8 - 4 = 4$ ft. which is the answer to (a).
Finally, setting $s = 0$ (return to ground) and solving for $t$, we have $0 = 16t - 16t^2$ so $t = 1$ sec. for (b).
If we know the formula for the velocity \( v(t) \) of a particle moving along a straight line, then between time \( t = t_0 \) and time \( t = t_1 \), we see that

\[
\int_{t_0}^{t_1} v(t) \, dt = \int_{t_0}^{t_1} s'(t) \, dt = s(t_1) - s(t_0)
\]

Thus the definite integral of the velocity between two times is the change in position that occurs during that time interval. This is called the \textit{displacement}.

In contrast, the total distance traveled in that time interval is given by

\[
d = \int_{t_0}^{t_1} |v(t)| \, dt
\]
Example. A particle moves on a coordinate line so that its velocity at any time \( t \) is
\[
v(t) = t^2 - 2t
\]
m/s.
(a) Find the displacement of the particle during the time interval \( 0 \leq t \leq 3 \).
(b) Find the distance traveled during the same time interval.

Solution. (a) The displacement is
\[
\int_{0}^{3} v(t) = \int_{0}^{3} (t^2 - 2t) dt = \left[ \frac{t^3}{3} - t^2 \right]_{0}^{3} = 9 - 9 = 0
\]
(b) We must integrate the absolute value by dividing the integral up into parts.

\[ \int_{0}^{3} |v(t)|\,dt = \int_{0}^{1} |v(t)|\,dt + \int_{1}^{3} |v(t)|\,dt = \int_{0}^{2} 2t - t^2\,dt + \int_{2}^{3} t^2 - 2t\,dt \]

\[ = \left[ t^2 - \frac{t^3}{3} \right]_{0}^{2} + \left[ \frac{t^3}{3} - t^2 \right]_{2}^{3} = \left( 4 - \frac{8}{3} \right) + \left( \frac{4}{3} - \frac{8}{3} \right) = \frac{8}{3} \]
The average value of a function

**Definition.** The average value of a function $f$ over an interval $[a, b]$ is defined by

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)dx$$
The average height times the base length is equal to the integral.
**Problem.** Find the average value of the function $e^x$ from $-1$ to $\ln 5$.

**Solution.**

$$f_{\text{ave}} = \frac{1}{(\ln(5)+1)} \int_{-1}^{\ln(5)} e^x \, dx = \frac{1}{(\ln(5)+1)} \left[ e^{\ln(5)} - e^{-1} \right] = \frac{5 - \frac{1}{e}}{(\ln(5)+1)}$$

**Problem.** Find the average value of the function $x^2$ from 1 to 4.

**Solution.**

$$f_{\text{ave}} = \frac{1}{4-1} \int_{1}^{4} x^2 \, dx = \frac{1}{3} \left[ \frac{x^3}{3} \right]_{1}^{4} = \frac{1}{3} \left[ \frac{64}{3} - \frac{1}{3} \right] = \frac{63}{9} = 7$$