Monotone Sequences

**Definition.** A sequence $\{a_n\}^{\infty}_{n=1}$ is called

(a) *strictly increasing* if $a_1 < a_2 < a_3 < ... < a_n < ...$

(b) *increasing* if $a_1 \leq a_2 \leq a_3 \leq ... \leq a_n \leq ...$

(c) *strictly decreasing* if $a_1 > a_2 > a_3 > ... > a_n > ...$

(d) *decreasing* if $a_1 \geq a_2 \geq a_3 \geq ... \geq a_n \geq ...$

A sequence that is either increasing or decreasing is called *monotone*. 
Strictly decreasing.
Strictly increasing.
Not monotone.
Tests for monotonicity.

(1) Take the difference between successive terms

(a) If \(a_{n+1} - a_n > 0\)  Strictly Increasing
(b) If \(a_{n+1} - a_n < 0\)  Strictly Decreasing
(c) If \(a_{n+1} - a_n \geq 0\)  Increasing
(d) If \(a_{n+1} - a_n \leq 0\)  Decreasing

(2) Take the ratio of Successive terms

(a) If \(\frac{a_{n+1}}{a_n} > 1\)  Strictly Increasing
(b) If \(\frac{a_{n+1}}{a_n} < 1\)  Strictly Decreasing
(c) If \(\frac{a_{n+1}}{a_n} \geq 1\)  Increasing
(d) If \(\frac{a_{n+1}}{a_n} \leq 1\)  Decreasing
Either test can be used, depending on the form of the terms of the sequence.

**Example.** Show that the sequence $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots$ is strictly increasing.

**Method 1.**

$$\frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)} = \frac{n^2 + 2n + 1 - (n^2 + 2n)}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)} > 0$$

Thus the sequence is strictly increasing.
Example. Show that the sequence \( \frac{1}{2}, 2, 3, \ldots, \frac{1}{n}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots \) is strictly increasing.

Method 2.

\[
\frac{n+1}{n+2} \cdot n = \frac{(n+1)^2}{n(n+2)} = \frac{n^2 + 2n + 1}{n^2 + 2n} > 1
\]

Again the sequence is strictly increasing.
**Example.** Determine whether the sequence

\[
\left\{1 - \frac{1}{n}\right\}_{n=1}^{\infty}
\]

is strictly increasing or strictly decreasing.

**Solution.** We use the method of successive differences.

\[
a_{n+1} - a_n = \left[1 - \frac{1}{n+1}\right] - \left[1 - \frac{1}{n}\right] = \left[\frac{1}{n} - \frac{1}{n+1}\right]
\]

\[
= \frac{(n+1) - n}{n(n+1)} = \frac{1}{n(n+1)} > 0
\]

Thus the sequence is strictly increasing.
Example. Determine whether the sequence

\[
\left\{ \frac{n}{4n-1} \right\}_{n=1}^{\infty}
\]

is strictly increasing or strictly decreasing.

Solution. We use the method of successive differences.

\[
a_{n+1} - a_n = \left[ \frac{n+1}{4n+3} \right] - \left[ \frac{n}{4n-1} \right] = \frac{(n+1)(4n-1) - n(4n+3)}{(4n-1)(4n+3)}
\]

\[
= \frac{(4n^2 + 3n - 1) - (4n^2 + 3n)}{(4n-1)(4n+3)} = \frac{-1}{(4n-1)(4n+3)} < 0
\]

Thus the sequence is strictly decreasing.
Example. Determine whether the sequence
\[ \left\{ n-n^2 \right\}_{n=1}^{\infty} \]
is strictly increasing or strictly decreasing.

Solution. We use the method of successive differences.

\[ a_{n+1} - a_n = \left[ (n+1)-(n+1)^2 \right] - \left[ n-n^2 \right] = -n^2 - n - n + n^2 = -2n < 0 \]

Thus the sequence is strictly decreasing.
Example. Determine whether the sequence
\[
\left\{ \frac{2^n}{1+2^n} \right\}_{n=1}^\infty
\]
is strictly increasing or strictly decreasing.

Solution. We use the method of successive ratios.

\[
a_{n+1}/a_n = \frac{2^{n+1}}{1+2^{n+1}} / \frac{2^n}{1+2^n} = \frac{2^{n+1}}{1+2^n} \cdot \frac{1+2^n}{1+2^{n+1}} = \frac{2^{n+1}+2^{2n+1}}{2^n+2^{2n+1}} > 1
\]

Thus the sequence is strictly increasing.
Example. Determine whether the sequence
\[
\left\{ \frac{10^n}{(2n)!} \right\}_{n=1}^\infty
\]
is strictly increasing or strictly decreasing.

Solution. We use the method of successive ratios.

\[
\frac{a_{n+1}}{a_n} = \frac{10^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{10^n} = \frac{10^{n+1}}{10^n} \cdot \frac{(2n)!}{(2n+2)!} = \frac{10}{(2n+1)(2n+2)} < 1
\]

Thus the sequence is strictly decreasing.
Example. Determine whether the sequence
\[
\left\{ \frac{5^n}{2(n^2)} \right\}_{n=1}^\infty
\]
is strictly increasing or strictly decreasing.

Solution. We use the method of successive ratios.

\[
a_{n+1}/a_n = \frac{5^{n+1}}{2(n+1)^2} / \frac{5^n}{2n^2} = \frac{5^{n+1}}{2(n+1)^2} \cdot \frac{2n^2}{2n^2} = \frac{5}{2(2n+1)} < 1
\]

Thus the sequence is strictly decreasing.
Another possible way to show that a sequence is monotone is in the case where \( a_n = f(n) \) and \( f(x) \) is a function whose derivative is always positive or always negative.
Example. Determine whether the sequence
\[ \left\{ 3 - \frac{1}{n} \right\}_{n=1}^{\infty} \]
is strictly increasing or strictly decreasing.

Solution.

Let \( f(x) = 3 - \frac{1}{x} \). Then \( f'(x) = \frac{1}{x^2} > 0 \).

Thus the function \( f \) is strictly increasing, so the sequence is strictly increasing.
Example. Determine whether the sequence

\[
\left\{ ne^{-2n} \right\}_{n=1}^{\infty}
\]

is strictly increasing or strictly decreasing.

Solution.

Let \( f(x) = xe^{-2x} \). Then \( f'(x) = e^{-2x} - 2xe^{-2x} = e^{-2x}(1 - 2x) < 0 \)
when \( x \geq 1 \).
Thus the function \( f \) is strictly decreasing when \( x \geq 1 \), so the sequence is strictly decreasing.
Example. Determine whether the sequence

\[ \left\{ \tan^{-1} n \right\}_{n=1}^{\infty} \]

is strictly increasing or strictly decreasing.

Solution.

Let \( f(x) = \tan^{-1}(x) \). Then \( f'(x) = \frac{1}{1+x^2} > 0 \)
when \( x \geq 1 \).

Thus the function \( f \) is strictly increasing when \( x \geq 1 \), so the sequence is strictly increasing.
Properties that hold eventually.

The first few terms of a sequence can be dropped without changing the limit. (The first few terms really means any finite number - thus you can drop the first 100 billion terms if you want, without changing the limit.)

**Example.** Show that the following sequences are eventually strictly increasing or eventually strictly decreasing.

(a) \( \left\{ n + \frac{17}{n} \right\}_{n=1}^\infty \)

(b) \( \left\{ n^5 e^{-n} \right\}_{n=1}^\infty \)
Example. Show that the following sequences are eventually strictly increasing or eventually strictly decreasing.

(a) \( \left\{ n + \frac{17}{n} \right\}_{n=1}^{\infty} \)  
(b) \( \left\{ n^5 e^{-n} \right\}_{n=1}^{\infty} \)

(a) Look at the function \( f(x) = x + \frac{17}{x} \). Then

\[
f'(x) = 1 - \frac{17}{x^2} = \frac{x^2 - 17}{x^2} > 0 \text{ when } x \geq 5.
\]

Thus this sequence is eventually strictly increasing, The first few terms are: 18, 10.5, 8.66, 8.25, 8.4, …
**Example.** Show that the following sequences are eventually strictly increasing or eventually strictly decreasing.

(a) \( \left\{ n + \frac{17}{n} \right\}_{n=1}^{\infty} \)

(b) \( \left\{ n^{5} e^{-n} \right\}_{n=1}^{\infty} \)

(b) Look at the function \( f(x) = x^{5} e^{-x} \). Then

\[ f'(x) = x^{4}(5-x)e^{-x} < 0 \text{ when } x \geq 6. \]

Thus this sequence is eventually strictly decreasing.
One of the most important things we need to be able to do is to see whether or not a sequence has a limit, even when we have little or no idea what that limit is.

This will be the usual situation when we get to infinite series. The definition of limit implies that we must know the limit $L$ before we prove that $L$ is the limit, but if we look at certain types of sequences, we can show that a limit exists without knowing what it is.
Completeness of the Real Numbers

**Theorem.** If a sequence is eventually increasing, then either
(1) The sequence has an *upper bound*, that is a constant $M$ so that $a_n \leq M$ for all $n$, in which case it has a limit, or

(2) The sequence diverges to $+\infty$, \[ \lim_{n \to \infty} a_n = +\infty. \]

**Theorem.** If a sequence is eventually decreasing, then either
(1) The sequence has a *lower bound*, that is a constant $M$ so that $M \leq a_n$ for all $n$, in which case it has a limit, or

(2) The sequence diverges to $-\infty$, \[ \lim_{n \to \infty} a_n = -\infty. \]
Problem. Define a sequence recursively by

\[ a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n} \quad \text{for } n \geq 1 \]

(a) Show that \( a_n < 2 \) for \( n \geq 1 \).
(b) Show that \( a_{n+1}^2 - a_n^2 = (2 - a_n)(1 + a_n) \) for \( n \geq 1 \).
(c) Show that the sequence converges and find its limit.

Solution. The first few terms are

\[ a_1 = \sqrt{2}, \quad a_2 = \sqrt{2 + \sqrt{2}}, \quad a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots \]

(a) If \( a_n < 2 \), then \( a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = 2 \).

Since \( a_1 < 2 \), it follows that \( a_2 < 2 \), which then implies that \( a_3 < 2 \), and so on. Formally this is a proof by mathematical induction.
Problem. Define a sequence recursively by

\[ a_1=\sqrt{2}, \quad a_{n+1}=\sqrt{2+a_n} \quad \text{for } n \geq 1 \]

(a) Show that \(a_n < 2\) for \(n \geq 1\).

(b) Show that \(a_{n+1}^2-a_n^2=(2-a_n)(1+a_n)\) for \(n \geq 1\)

(c) Show that the sequence converges and find its limit.

Solution.

(b) \(a_{n+1}^2=2+a_n\) so

\[ a_{n+1}^2-a_n^2=(a_{n+1}^2-a_n^2)=(a_n^2-a_n-2)=-(a_n-2)(a_n+1)=(2-a_n)(a_n+1) \]

Since this expression is positive for all \(n \geq 1\), the sequence is increasing.
Problem. Define a sequence recursively by

\[ a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n} \quad \text{for } n \geq 1 \]

(a) Show that \( a_n < 2 \) for \( n \geq 1 \).

(b) Show that \( a_{n+1}^2 - a_n^2 = (2 - a_n)(1 + a_n) \) for \( n \geq 1 \).

(c) Show that the sequence converges and find its limit.

Solution.

(c) Since the sequence is increasing and has an upper bound, it must converge (to a limit \( L \) somewhere between 2 and \( \sqrt{2} \)).

Passing to the limit in the definition of the sequence, we see that:

\[ L = \sqrt{2 + L} ; \quad L^2 - L - 2 = (L - 2)(L + 1) = 0. \]

Since \( L \) cannot be \(-1\), it must be 2.
There is no rational number whose square is 2. Assume first that any common factors of $p$ and $q$ have been cancelled. If $(p/q)^2 = 2$, then $p^2 = 2q^2$ so $p$ is even, say $p = 2r$. Then $p^2 = 4r^2 = 2q^2$ so $q^2 = 2r^2$ and $q$ is even. However this is impossible, since then $p$ and $q$ would have a common factor of 2.
This shows that there are sequences of rational numbers that are monotone and bounded, such as successive decimal approximations to the square root of two, but that do not have a limit in the rational numbers