Infinite Series

**Definition.** An *infinite series* is an expression of the form

\[
\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots \tag{1}
\]

Where the numbers \( u_k \) are called the *terms* of the series.

Such an expression is meant to be the result of adding together infinitely many numbers. As with improper integrals, the above expression can sometimes be given a meaning and sometimes not.
Since it is physically impossible to add together an infinite number of terms, we must attempt to give expression (1) meaning through the idea of limit.

Infinite series are used in ordinary arithmetic in the form of decimal expansions. For example when we write $2/3 = 0.6666666\ldots$ we are expressing the idea that the infinite series

$$0.666\ldots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \ldots + \frac{6}{10^k} + \ldots$$

is somehow meaningful and has the value $2/3$. 
How might we give meaning to a series? One way is to begin by adding together all the terms of the series from term 1 to term $n$. The result is

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n = \sum_{k=1}^{n} u_k$$

and is called the $\textit{nth partial sum}$ of the series

$$\sum_{k=1}^{\infty} u_k$$

We then focus attention on the sequence of partial sums

$$\left\{s_n\right\}_{n=1}^{\infty} = s_1, s_2, s_3, \ldots, s_n, \ldots$$

$$= \left[u_1\right], \left[u_1 + u_2\right], \left[u_1 + u_2 + u_3\right], \ldots, \left[u_1 + u_2 + \cdots + u_n\right], \ldots$$
As $n$ gets ever larger, the partial sum $s_n$ represents the sum of more and more terms of the series. Thus it is reasonable to think that if the sequence $s_n$ tends to a limit $u$ (and therefore gets ever closer to $u$) the limit $u$ is what we should mean by the sum of all the terms. Thus we are led to the following definition.

**Definition.** Suppose that $\sum_{k=1}^{\infty} u_k$ is a series and that $\{s_n\}_{n=1}^{\infty}$ is the corresponding sequence of partial sums. If the sequence $\{s_n\}_{n=1}^{\infty}$ converges to a limit $s$, we say that the series $\sum_{k=1}^{\infty} u_k$ converges to $s$ and that $s$ is the sum of the series. If the sequence of partial sums diverges, we say the series diverges, and therefore has no sum.
**Example.** Determine whether the series

\[ 1 - 1 + 1 - 1 + \cdots \]

converges or diverges, and if it converges, find its sum.

**Solution.** We see that \( s_1 = 1, \ s_2 = 0, \ s_3 = 1, \ s_4 = 0, \ldots \)

Since this sequence oscillates between 0 and 1, it will not eventually get and stay close to any number, and so it has no limit.

Thus the series \( 1 - 1 + 1 - 1 + \cdots \) does not converge and so has no sum.
Without a formal definition, this sequence could, and did, cause controversy.

On one hand you can reason that the sum should be

\[(1-1)+(1-1)+\cdots=0\]

On the other hand we could write the series as

\[1+(-1+1)+(-1+1)+\cdots=1\]

The definition shows us that no meaning at all can be attached to the series.
Geometric Series

One special type of series occurs frequently and is easily handled. That is the series

\[ a + ar + ar^2 + ar^3 + \cdots + ar^k + \cdots = \sum_{k=0}^{\infty} ar^k \]

This is called a geometric series, and \( r \) is called the ratio. Each term is multiplied by \( r \) to produce the next term.

Note here that we have begun the series with a subscript 0 rather than 1, and so started with the 0th term instead of the first term. Actually we can begin the series with any subscript, just as was the case with a sequence, but 0 and 1 are the most common starting points..
The geometric series is handled by a simple algebraic trick.  

The $n$th partial sum of the series $\sum_{k=0}^{\infty} ar^k$ is 

$$s_n = a + ar + ar^2 + ar^3 \ldots + ar^n$$

then $r s_n = ar + ar^2 + ar^3 \ldots + ar^n + ar^{n+1}$ and therefore 

$$s_n - r s_n = [a + ar + ar^2 \ldots + ar^n] - ar + ar^2 + ar^3 \ldots + ar^n + ar^{n+1} = a - ar^{n+1} = a(1-r^{n+1}).$$

We have 

$$s_n = a \left[ \frac{1-r^{n+1}}{1-r} \right] \text{ if } r \neq 1$$

(it is easily seen that the series diverges if $r = 1$ or $r = -1$)
If $|r| > 1$, $s_n$ does not converge. If $|r| < 1$, $s_n$ does converge since powers of $r$ tend to 0. We therefore have

If $|r| \geq 1$, the series $\sum_{k=0}^{\infty} ar^k$ diverges. If $|r| < 1$, the series converges and

$$s = \lim_{n \to \infty} s = a \lim_{n \to \infty} \left[ \frac{1 - r^{n+1}}{1 - r} \right] = \frac{a}{1 - r}.$$ 

Therefore $s$ is the sum of the series.
Example. Determine whether the series
\[ \sum_{k=1}^{\infty} \left( \frac{2}{3} \right)^{k+2} = \left( \frac{2}{3} \right)^{3} + \left( \frac{2}{3} \right)^{4} + \left( \frac{2}{3} \right)^{5} + \cdots = \frac{8}{27} + \frac{16}{81} + \frac{32}{243} + \cdots \]
converges or diverges, and if it converges determine the sum.

Solution. This is a geometric series, since each term is multiplied by 2/3 to get the next. We have \( a = \frac{8}{27} \) and \( r = \frac{2}{3} \). Since the ratio has absolute value less than 1, the series converges to
\[
\frac{a}{1-r} = \frac{\frac{8}{27}}{1-\frac{2}{3}} = \frac{8}{9}
\]
Example. Determine whether the series
\[
\sum_{k=1}^{\infty} \left( -\frac{3}{2} \right)^{k+1} = \left( -\frac{3}{2} \right)^{2} + \left( -\frac{3}{2} \right)^{3} + \left( -\frac{3}{2} \right)^{4} + \cdots = \frac{9}{4} - \frac{27}{8} + \frac{81}{16} - \cdots
\]
converges or diverges, and if it converges determine the sum.

Solution. This is a geometric series, since each term is multiplied by 2/3 to get the next. We have \( a = 9/4 \) and \( r = -3/2 \). Since the ratio has absolute value greater than 1, the series diverges.
Example. Determine whether the series converges or diverges, and if it converges determine the sum.

\[
\sum_{k=5}^{\infty} \left( \frac{e}{\pi} \right)^{k-1} = \left( \frac{e}{\pi} \right)^4 + \left( \frac{e}{\pi} \right)^5 + \left( \frac{e}{\pi} \right)^6 + \cdots
\]

converges or diverges, and if it converges determine the sum.

Solution. This is a geometric series, since each term is multiplied by \( e/\pi \) to get the next. We have \( a = e^4/ \pi^4 \) and \( r = e/\pi \). Since the ratio has absolute value less than 1, the series converges to

\[
\frac{a}{1-r} = \frac{e^4}{\pi^4} \frac{1}{1 - \frac{e}{\pi}} = \frac{e^4}{\pi^4} \frac{\pi}{\pi - e} = \frac{e^4}{\pi^3 (\pi - e)}.
\]
Example. Determine whether the series

\[ \sum_{k=1}^{\infty} 5^{3k} 7^{-1-k} = 7 \sum_{k=1}^{\infty} \left( \frac{625}{7} \right)^k \]

converges or diverges, and if it converges determine the sum.

Solution. This is a geometric series, since each term is multiplied by the ratio 625/7 to get the next. Since the ratio has absolute value greater than 1, the series diverges.
Every decimal that eventually repeats a string of digits endlessly is said to be repeating. Every rational number can be expressed as a repeating decimal, and conversely every repeating decimal converges to a rational number. Examples of repeating decimals are.

.333333333… 2.456666666666666…
12.472345345345345… .00131513151315…
0.5555555… 0.9999999…
Problem. Find the rational number that is the sum of the repeating decimal

1.333333333…

Solution. This decimal is the same as the infinite series

\[ 1 + \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} \cdots = 1 + \sum_{k=0}^{\infty} \frac{3}{10} \left( \frac{1}{10} \right)^k \]

The series that follows the 1 is geometric with ratio 1/10. Thus it converges to the number

\[ \frac{3}{10} = \frac{1}{3} \]

\[ 1 - \frac{1}{10} \]

The decimal therefore represents 4/3.
**Problem.** Find the rational number that is the sum of the repeating decimal

\[0.99999\ldots\]

**Solution.** This decimal is the same as the infinite series

\[
\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \ldots = \sum_{k=0}^{\infty} \frac{9}{10} \left[ \frac{1}{10} \right]^k
\]

The series is geometric with ratio 1/10. Thus it converges to the number

\[
\frac{9}{10 - \frac{1}{10}} = 1
\]

The decimal therefore represents 1.
Problem. Find the rational number that is the sum of the repeating decimal

\[ 0.159159159\ldots \]

Solution. This decimal is the same as the infinite series

\[
\frac{159}{10^3} + \frac{159}{10^6} + \frac{159}{10^9} + \ldots = \sum_{k=0}^{\infty} \frac{159}{10^3} \left[ \frac{1}{10^3} \right]^k
\]

The series is geometric with ratio \(1/1000\). Thus it converges to the number

\[
\frac{\frac{159}{1000}}{1 - \frac{1}{1000}} = \frac{159}{1000 - 1}
\]

The decimal therefore represents \(159/999\).
Problem. Find the rational number that is the sum of the repeating decimal

$$0.212121\ldots$$

Solution. This decimal is the same as the infinite series

$$\frac{21}{10^2} + \frac{21}{10^4} + \frac{21}{10^6} + \ldots = \sum_{k=0}^{\infty} \frac{21}{10^2} \left[ \frac{1}{10^2} \right]^k$$

The series is geometric with ratio $1/1000$. Thus it converges to the number

$$\frac{21}{100} \left[ 1 - \frac{1}{100} \right] = \frac{21}{99}$$

The decimal therefore represents $21/99$. 
For most series, we will try to determine convergence or divergence, but we will not be able to find the sum. The geometric series is one exception. Another is the so-called telescoping series. The following example illustrates this case.

**Example.** Find the sum of the series

\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \ldots + \frac{1}{k \cdot (k+1)} + \ldots
\]

**Solution.** The technique of partial fractions shows that

\[
\frac{1}{k \cdot (k+1)} = \left[ \frac{1}{k} - \frac{1}{(k+1)} \right]
\]
Thus we can rewrite the $n^{th}$ partial sum of the series as

$$s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot n+1} = \left[ 1 - \frac{1}{2} \right] + \left[ \frac{1}{2} - \frac{1}{3} \right] + \left[ \frac{1}{3} - \frac{1}{4} \right] + \cdots + \left[ \frac{1}{n} - \frac{1}{n+1} \right]$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Then

$$\lim_{n \to \infty} s_n = 1$$

so

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{k \cdot k+1} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k \cdot k+1} = 1$$
Once it was believed that if the terms of a series tended to 0, then the series converged. This is false, as the following example shows.

\[
\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \infty
\]

\[
s_2 = 1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = 1
\]

\[
s_4 = s_2 + \frac{1}{3} + \frac{1}{4} > s_2 + \frac{1}{4} + \frac{1}{4} > \frac{3}{2}
\]

\[
s_8 = s_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > s_4 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} > \frac{4}{2}
\]

\[
s_{16} > \frac{5}{2}
\]

and in general \( s_{2^n} > \frac{n+1}{2} \).

On the other hand, \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \) converges.
Problem. Find all values of $x$ for which the series
\[ \sum_{k=0}^{\infty} (x-3)^{k} = 1 + (x-3) + (x-3)^2 + (x-3)^3 + \ldots \]
converges, and find the sum of the series for those values of $x$.

The series is geometric, and the ratio is $x - 3$. Thus we know that it converges when $|x - 3| < 1$. This is when $-1 < x - 3 < 1$ or $2 < x < 4$.

When it converges, it converges to
\[ \frac{1}{1-(x-3)} = \frac{1}{4-x} \]
Problem. Find all values of \( x \) for which the series

\[
\sum_{k=1}^{\infty} \left( \frac{3^k}{x^k} \right) = \frac{3}{x} + \frac{9}{x^2} + \frac{27}{x^3} + \ldots
\]

converges, and find the sum of the series for those values of \( x \).

The series is geometric, and the ratio is \( 3/x \). Thus we know that it converges when \( |3/x| < 1 \). This is when \( x > 3 \) or \( x < -3 \).

This happens when \( x/3 > 1 \) or \( x/3 < -1 \), or when \( x > 3 \) or \( x < -3 \).

When it converges, it converges to

\[
\frac{3}{1 - \frac{3}{x}} = \frac{3}{x - 3}
\]