Tangent Lines and Arc Length for Parametric and Polar Curves.

In earlier lectures we looked at parametric curves, that is those that can be written as

\[ x = f(t); y = g(t); a \leq t \leq b \]

Then it can be shown that

\[ \frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} \]

This allows us to find the slope of tangent lines without eliminating the parameter.
Example. Find the tangent line to the curve 
\[ x = t - 2\sin t; \quad y = 2 - \cos t; \quad t \geq 0 \]
at the point where \( t = 9\pi/4 \).

We have
\[
\frac{dx}{dt} = 1 - 2\cos t = 1 - \sqrt{2} =
\]
\[
\frac{dy}{dt} = \sin t = \frac{1}{\sqrt{2}}
\]

Thus
\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{\sqrt{2}} / \left(1 - \sqrt{2}\right) = \frac{1}{\sqrt{2} \left(1 - \sqrt{2}\right)} = \frac{1}{\sqrt{2} - 2} \approx -1.7
\]

This situation is shown below.
Problem. Find $dy/dx$ and $d^2y/dx^2$ at the point on the curve $x = \frac{1}{2}t^2, y = \frac{1}{3}t^3$ where $t = 2$.

Solution. \[
\frac{dx}{dt} = t, \quad \frac{dy}{dt} = t^2 \quad \text{Therefore} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{t^2}{t} = t
\]

At the point where $t = 2$, $dy/dx = 2$.

Now if we let $y' = dy/dx = t$, then

\[
\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{1}{t}
\]

At the point where $t = 2$, $d^2y/dx^2 = 1/2$. 
**Problem.** Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point on the curve

$$x = \cos \varphi; \ y = 3 \sin \varphi$$

where $\varphi = 5\pi/6$

**Solution.**

\[
\frac{dx}{d\varphi} = -\sin \varphi \quad \frac{dy}{d\varphi} = 3 \cos \varphi
\]

Therefore

\[
\frac{dy}{dx} = \frac{dy}{d\varphi} \times \frac{1}{dx} = \frac{3 \cos \varphi}{-\sin \varphi} = -3 \cot \varphi
\]

At the point where $\varphi = 5\pi/6$, $\frac{dy}{dx} = 3\sqrt{3}$.

Now if we let $y' = \frac{dy}{dx} = -3 \cot \varphi$, then

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy'}{dx} \right) = \frac{d}{dx} \left( \frac{dy}{d\varphi} \times \frac{1}{dx} \right) = \frac{d}{dx} \left( \frac{3 \cos \varphi}{-\sin \varphi} \right)
\]

\[
= -3 \csc^2 \varphi - \sin \varphi
\]

At the point where $\varphi = 5\pi/6$, $\frac{d^2y}{dx^2} = -24$. 


Problem. Find all values of $t$ at which the parametric curve

$$x=2t^3-15t^2+24t+7; y=t^2+t+1$$

has (a) a horizontal tangent line and (b) a vertical tangent line.

Solution. \[
\frac{dx}{dt}=6t^2-30t+24=6(t^2-5t+4)=6(t-4)(t-1)
\]

\[
\frac{dy}{dt}=2t+1 \quad \text{Thus} \quad \frac{dy}{dx} = \frac{2t+1}{6(t-4)(t-1)}
\]

We see that the tangent line is horizontal when $2t + 1 = 0$, or $t = -1/2$. The tangent line is vertical when $t = 4$ or $t = 1$. 
We can apply this procedure to find the tangent line to a polar curve. A polar curve \( r = f(\theta) \) can be written parametrically by noting that \( x = r\cos(\theta) \) and \( y = r\sin(\theta) \). Thus

\[
x = f(\theta)\cos\theta \\
y = f(\theta)\sin\theta
\]

This means that

\[
\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta = -r\sin\theta + \frac{dr}{d\theta}\cos\theta \\
\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta = r\cos\theta + \frac{dr}{d\theta}\sin\theta
\]

Therefore at any point we have

\[
\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta \frac{dr}{d\theta}}{-r\sin\theta + \cos\theta \frac{dr}{d\theta}}
\]
Problem. Find the slope of the tangent line to the polar curve for $r = 1 + \sin \theta$ for $\theta = \pi/4$.

Solution. $dr/d\theta = \cos \theta$. Thus

$$\frac{dy}{dx} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}} = \frac{(1+\sin \theta) \cos \theta + \sin \theta \cos \theta}{-(1+\sin \theta) \sin \theta + \cos \theta \cos \theta}$$

$$\frac{dy}{dx} = \frac{(1+\frac{1}{\sqrt{2}}) \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}}{-(1+\frac{1}{\sqrt{2}}) \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}} = \frac{1+\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -\sqrt{2} - 1$$
**Problem.** Find the slope of the tangent line to the polar curve  

\[ r = a \sec 2\theta \]

for \( \theta = \pi/6 \).

**Solution.** \( dr/d\theta = 2a \sec 2\theta \tan 2\theta \). Thus at \( \theta = \pi/6 \), we have  

\[ dr/d\theta = 2a \sec 2\theta \tan 2\theta = 4a\sqrt{3}, \quad r = 2a, \quad \sin \theta = 1/2, \quad \cos \theta = \sqrt{3}/2. \]

Thus  

\[ \frac{dy}{dx} = \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}} = \frac{2a \left( \frac{\sqrt{3}}{2} \right) + \frac{1}{2} 4a\sqrt{3}}{-2a \left( \frac{1}{2} \right) + \left( \frac{\sqrt{3}}{2} \right) 4a\sqrt{3}} \]

\[ = \frac{3a\sqrt{3}}{5a} = \frac{3\sqrt{3}}{5} \]
We can find out some useful information about polar curves that pass through the pole.

**Theorem.** Suppose that the curve $r = f(\theta)$ passes through the pole, or origin, at $\theta = \theta_0$. If $dr/d\theta$ is not 0 at $\theta = \theta_0$, then the line $\theta = \theta_0$ is the tangent line to the curve at $\theta = \theta_0$ at the origin.

**Proof.**

\[
\begin{align*}
\frac{dy}{dx} &= \frac{r \cos \theta + \sin \theta \frac{dr}{d\theta}}{-r \sin \theta + \cos \theta \frac{dr}{d\theta}} = \frac{\sin \theta \frac{dr}{d\theta}}{\cos \theta \frac{dr}{d\theta}} = \tan \theta \\
\end{align*}
\]

at the origin, under the conditions given.
Problem. Sketch the polar curve

\[ r = 4 \cos \theta \]

and find the polar equations of the tangent line to the curve at the pole.

Solution. The curve can be written as \( r^2 = 4r \cos \theta \)

or \( x^2 + y^2 = 4x \). On completing the square, this becomes

\[ (x-2)^2 + y^2 = 4. \]

The graph is:

Since at \( \theta = \pi/2 \), \( r = 0 \) and \( dr/d\theta = -4\sin \theta = -4 \), the previous theorem tells us that the tangent line at the pole is \( \theta = \pi/2 \).
**Problem.** Sketch the polar curve

\[ r = \sin 2\theta \]

and find the polar equations of the tangent line to the curve at the pole.

**Solution.** The curve is a 4 leaf rose whose graph is:

We have \( r = 0 \) when \( \theta = 0 \), and \( \theta = \pi/2 \). At these points, the derivative is \( 2\cos 2\theta = 1 \) or \( -1 \). Thus the tangent lines at the pole have equations \( \theta = 0 \) and \( \theta = \pi/2 \), by the previous theorem.
Recall from earlier lectures that the differential element of arc length, $ds$ can be represented by the diagram

$$ds = \sqrt{dx^2 + dy^2}$$
Then arc length can be found by integration,

\[ s = \int ds = \int \sqrt{dx^2 + dy^2} \]

We previously investigated various forms of this equation, corresponding to \( y = f(x), \ x = g(y), \) and the parametric curve \( x = x(t), \ y = y(t). \) The parametric form follows from the differential substitutions

\[ dx = \frac{dx}{dt} dt \quad \quad dy = \frac{dy}{dt} dt \]

so that

\[ s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]
We apply this formula to find a formula for arc length of a polar curve. We know that a parametric form of the polar curve \( r = f(\theta) \) is given by

\[
x = f(\theta) \cos(\theta); \quad y = f(\theta) \sin(\theta)
\]

Then

\[
\frac{dx}{d\theta} = -f(\theta) \sin(\theta) + f'(\theta) \cos(\theta) = -r \sin(\theta) + \frac{dr}{d\theta} \cos(\theta)
\]

\[
\frac{dy}{d\theta} = f(\theta) \cos(\theta) + f'(\theta) \sin(\theta) = r \cos(\theta) + \frac{dr}{d\theta} \sin(\theta)
\]

By squaring these expressions, we get
We add these equations to get:

\[
\left( \frac{dx}{d\theta} \right)^2 = r^2 \sin^2(\theta) - 2r \left( \frac{dr}{d\theta} \right) \sin(\theta) \cos(\theta) + \left( \frac{dr}{d\theta} \right)^2 \cos^2(\theta)
\]

\[
\left( \frac{dy}{d\theta} \right)^2 = r^2 \cos^2(\theta) + 2r \left( \frac{dr}{d\theta} \right) \cos(\theta) \sin(\theta) + \left( \frac{dr}{d\theta} \right)^2 \sin^2(\theta)
\]

We add these equations to get:

\[
\left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2 = \left[ r^2 + \left( \frac{dr}{d\theta} \right)^2 \right] \left[ \cos^2(\theta) + \sin^2(\theta) \right]
\]

\[
= r^2 + \left( \frac{dr}{d\theta} \right)^2
\]
Thus we finally arrive at the formula for polar arc length.

\[
 s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta
\]
**Problem.** Find the length of the spiral $r = e^\theta$ between $\theta = 0$ and $\theta = \pi/2$.

This curve is as shown below.

Using the arc length formula, we have

$$s = \frac{\pi}{2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta = \frac{\pi}{2} \sqrt{(e^\theta)^2 + (e^\theta)^2} d\theta = \sqrt{2} \frac{\pi}{2} e^\theta d\theta$$

$$= \sqrt{2}(e^{\pi/2} - 1) \approx 31.3$$
Problem. Find the length of the entire circle \( r = 2a \cos(\theta) \).

\[
\frac{dr}{d\theta} = -2a \sin(\theta) \quad \text{so} \quad r^2 + \left( \frac{dr}{d\theta} \right)^2 = 4a^2 \left[ \cos^2(\theta) + \sin^2(\theta) \right] = 4a^2
\]

\[
s = \int_{-\pi/2}^{\pi/2} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta = \int_{-\pi/2}^{\pi/2} 2a \, \theta = 2\pi a.
\]
**Problem.** Find the length of the polar curve \( r = \sin^2(\theta/2) \) from \( \theta = 0 \) to \( \theta = \pi \).

\[
\frac{dr}{d\theta} = \sin(\theta/2) \cos(\theta/2) \quad \text{so}
\]

\[
r^2 + \left( \frac{dr}{d\theta} \right)^2 = \sin^4(\theta/2) + \sin^2(\theta/2) \cos^2(\theta/2) = \sin^2(\theta/2)
\]

\[
s = \int_0^\pi \sin(\theta/2) \, d\theta = -2 \cos(\theta/2) \bigg|_0^\pi = 2
\]
Problem. Find the length of the polar curve \( r = \sin^3(\theta/3) \)
from \( \theta = 0 \) to \( \theta = \pi \).

\[
\frac{dr}{d\theta} = \sin^2(\theta/3) \cos(\theta/3) \text{ so }
\]

\[
r^2 + \left(\frac{dr}{d\theta}\right)^2 = \sin^6(\theta/3) + \sin^4(\theta/3) \cos^2(\theta/3) = \sin^4(\theta/3)
\]

\[
s = \frac{\pi}{2} \int_0^{\pi/2} \sin^2(\theta/3) d\theta = \frac{1}{2} \int_0^{\pi/2} \left[ 1 - \cos(2\theta/3) \right] d\theta = \frac{1}{2} \left[ \theta - \frac{3}{2} \sin(2\theta/3) \right]_0^{\pi/2}
\]

\[
= \frac{1}{2} \left[ \frac{\pi}{2} - \frac{3}{2} \sin\left(\frac{\pi}{3}\right) \right] = \frac{\pi}{4} - \frac{3\sqrt{3}}{8}
\]