Improper Integrals

In the definition of the definite integral

\[ \int_{a}^{b} f(x) \, dx \]

Two things are essential - without them the integral cannot exist - they are

1. The interval of integration is a finite interval \([a, b]\) (the definition would also work for \((a, b), (a, b], [a, b)\)).

2. The function \(f\) is bounded on the interval.

(There are also other, more technical conditions required of \(f\) for the integral to exist, but we won’t discuss them here.)
If either of these conditions is violated, then the integral is not defined, and we call it an **improper integral**. Some improper integrals can be given meaning and some cannot. We will investigate first the possible meaning of integrals that violate condition 1, that is integrals over infinite intervals.

The following are improper integrals because they violate condition 1. As yet they have no meaning. Some never will.

\[
\int_{1}^{\infty} \frac{dx}{x^2} \quad \int_{-\infty}^{1} \frac{dx}{x} \quad \int_{4}^{\infty} e^{-x} dx \quad \int_{1}^{\infty} \cos(x) dx
\]
The key to giving meaning to improper integrals is the concept of limit. Consider the expression
\[ \int_{a}^{\infty} f(x) \, dx \]
Where \( f \) is continuous and nonnegative in the interval \( (a, \infty) \). In keeping with the meaning of finite integrals, we may think of this integral as representing the area under the graph of \( f \) over the interval \([a, \infty)\), This is illustrated in the diagram below.
Of course this intuitive idea doesn’t help computationally, since we don’t know how to compute such an area (it seems as though it should be infinite - but it is not always so) except by integration, and we can’t integrate over an infinite interval.

Suppose that we first compute the integral (the area) from the starting point $a$ to some temporarily fixed point $l$ lying to the right of $a$. 
This is illustrated in the diagram above.

It is intuitively clear that if we now regard $l$ as a variable quantity, and slide it to the right, the corresponding integral (area) would get ever closer to the complete area under the curve. This leads us to make the following definition.
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It is intuitively clear that if we now regard \( l \) as a variable quantity, and slide it to the right, the corresponding integral (area) would get ever closer to the complete area under the curve. This leads us to make the following definition.

**Definition.** The improper integral of \( f \) over \([a, +\infty)\) is defined as

\[
\int_{a}^{\infty} f(x)dx = \lim_{l \to \infty} \int_{a}^{l} f(x)dx
\]
If this limit exists and is the number $L$, then we say that the integral
\[ \int_{a}^{\infty} f(x)\,dx \]
\[ \text{converges to } L, \text{ or that } L \text{ is the value of the integral.} \]

If the integral does not converge, we say that it \textit{diverges}. This can happen in two ways.

1. In this case we say that the integral \[ \lim_{l \to \infty} \int_{a}^{l} f(x)\,dx = \infty \text{ or } -\infty \] In this case we say that the integral diverges to plus infinity or minus infinity. The integral does not converge or have an actual value, but it behaves in a predictable way. We then write
\[ \int_{a}^{\infty} f(x) \, dx = -\infty \quad \text{or} \quad \int_{a}^{\infty} f(x) \, dx = \infty \] in these cases.

2. The limit can simply fail to exist in any sense, for example the expression involving \( l \) can oscillate back and forth between different values. There is no notation for this case, and we simply say that the integral diverges.
Example. Evaluate \[ \int_{2}^{\infty} \frac{dx}{x^2} \]

Solution. First compute \[ \int_{2}^{l} \frac{dx}{x^2} \] where \( l \) is a variable point greater than 2.

\[ \int_{2}^{l} \frac{dx}{x^2} = - \frac{1}{x} \bigg|_{2}^{l} = 1 - \frac{1}{l} \]

Thus \[ \int_{2}^{\infty} \frac{dx}{x^2} = \lim_{l \to \infty} \left( 1 - \frac{1}{l} \right) = 1 \]

This integral converges to 1/2. In this case we can also say that the area under the curve is 1/2, or that the value of the integral is 1/2. Thus an area may be finite even if it extends infinitely far in one direction.
Example. Evaluate $\int_{\frac{2}{x}}^{\infty} dx$

Solution. First compute $\int_{\frac{2}{x}}^{\infty} dx$ where $l$ is a variable point greater than 2.

$$\int_{\frac{2}{x}}^{\infty} dx = \lim_{l \to \infty} \ln(l) - \ln(2)$$

Thus $\int_{\frac{2}{x}}^{\infty} dx = \lim_{l \to \infty} \ln(l) - \ln(2) = \infty$

This integral diverges to $+\infty$ and we write $\int_{\frac{2}{x}}^{\infty} dx = \infty$

The integral does not converge, and the value does not exist as a real number, but the behavior is specific and is indicated by the notation. Here we also say that the area under the curve is infinite.
Note that the two graphs look similar, so it is not possible to tell by inspection when an area over an infinite interval is finite.

\[ y = \frac{1}{x^2} \]

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Example. Evaluate \( \int_0^\infty \cos(x)dx \)

Solution. First compute \( \int_0^l \cos(x)dx \) where \( l \) is a variable point greater than 1.

\[
\int_0^l \cos(x)dx = \sin(x) \bigg|_0^l = \sin(l) - \sin(0) = \sin(l)
\]

Thus we need to consider whether \( \lim_{l \to \infty} \sin(l) \) exists or not. This is illustrated in the following diagram.
The question is, when $l$ moves to the right arbitrarily far, does the height tend to a fixed limit. The answer is no. The height continually oscillates between values. Thus the limit does not exist, nor does the quantity tend to plus or minus infinity. There is no notation to use here. We simply say that the integral 

\[ \int_{0}^{\infty} \cos(x) \, dx \]

diverges, or that it does not have a value.
Example. Evaluate $\int_{0}^{\infty} xe^{-x} \, dx$

Solution. First compute $\int_{0}^{l} xe^{-x} \, dx$ where $l$ is a variable point greater than 0.

This has to be done using integration by parts.

In general, with an improper integral, we may need to use any method of integration that we have studied, and then any method of finding limits that we have studied.

Let $u = x$ and $dv = e^{-x} \, dx$. Then $du = dx$ and $v = -e^{-x}$.

$$\begin{align*}
\int_{0}^{l} xe^{-x} \, dx &= -xe^{-x} \bigg|_{0}^{l} + \int_{0}^{l} e^{-x} \, dx = -le^{-l} - e^{-l} \bigg|_{0}^{l} = 1 - le^{-l} - e^{-l}
\end{align*}$$
Then \[ \int_{0}^{\infty} xe^{-x} \, dx = \lim_{l \to \infty} \left[ 1 - le^{-l} - e^{-l} \right] = 1 \]

We see that the area under the curve shown below is 1.
In a similar way we can define the related integrals

\[ \int_{-\infty}^{b} f(x)dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)dx \]

**Definition.** The improper integral of \( f \) over \((-\infty, b] \) is defined as

\[ \int_{-\infty}^{b} f(x)dx = \lim_{l \to -\infty} \int_{l}^{b} f(x)dx \]

**Definition.** The improper integral of \( f \) over \((-\infty, +\infty) \) is defined as

\[ \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx \]

where \( c \) is any real number.
**Example.** Evaluate \[ \int_{-\infty}^{0} \frac{e^x}{3-2e^x} \, dx \]

**Solution.** First compute \[ \int_{l}^{0} \frac{e^x}{3-2e^x} \, dx \] where \( l \) is a variable point less than 0.

Let \( u = 3 - 2e^x \), \( du = -2e^x \, dx \). Then

\[
\int \frac{e^x}{3-2e^x} \, dx = -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln|u| = -\frac{1}{2} \ln|3-2e^x|
\]

so

\[
\int_{l}^{0} \frac{e^x}{3-2e^x} \, dx = -\frac{1}{2} \ln|3-2e^x| \bigg|_{l}^{0} = \frac{1}{2} \ln|3-2e^l|
\]
\[ \int_{-\infty}^{0} \frac{e^x}{3-2e^x} \, dx = \lim_{l \to -\infty} \frac{1}{2} \ln \left| \frac{3-2e^l}{2} \right| = \frac{\ln 3}{2} \]
**Example.** Evaluate \( \int_{\infty}^{\infty} \frac{dx}{1+x^2} \)

**Solution.** We will compute this as \( \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} \)

\[
\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{l \to -\infty} \int_{l}^{0} \frac{dx}{1+x^2} = \lim_{l \to -\infty} \tan^{-1}(x) \Bigg|_{l}^{0} = \lim_{l \to -\infty} \left[ -\tan^{-1}(l) \right] = \frac{\pi}{2}
\]
\[
\int_0^\infty \frac{dx}{1+x^2} = \lim_{l \to \infty} \int_0^l \frac{dx}{1+x^2} = \lim_{l \to \infty} \tan^{-1}(x) \bigg|_0^l = \lim_{l \to \infty} \tan^{-1}(l) = \frac{\pi}{2}
\]

so
\[
\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi
\]
You might think that we could define the integral from minus infinity to plus infinity as follows:

\[ \int_{-\infty}^{\infty} f(x)dx = \lim_{l \to \infty} \int_{-l}^{l} f(x)dx \]

This does not work, as we see in the following example.

\[ \int_{-\infty}^{\infty} xdx = \lim_{l \to \infty} \int_{-l}^{l} xdx = \frac{l^2}{2} - \frac{l^2}{2} = 0 \]
Actually, it is easy to see that

\[
\int_{-\infty}^{0} x \, dx = \lim_{l \to \infty} \int_{-l}^{0} x \, dx = \lim_{l \to \infty} -\frac{l^2}{2} = -\infty
\]

and

\[
\int_{0}^{\infty} x \, dx = \lim_{l \to \infty} \int_{0}^{l} x \, dx = \lim_{l \to \infty} \frac{l^2}{2} = \infty
\]

Thus the integral \( \int_{-\infty}^{\infty} f(x) \, dx \) does not exist.