First Order Differential Equations and Applications

**Definition.** A differential Equation is an equation involving an unknown function \( y \), and some of its derivatives, and possibly some other known functions.

**Examples.**

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
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<tbody>
<tr>
<td>(1) ( \frac{dy}{dx} = \cos(x) )</td>
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<tr>
<td>(2) ( \frac{dy}{dx} = y + x^2 )</td>
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<tr>
<td>(3) ( \frac{d^2y}{dx^2} + e^x \frac{dy}{dx} + 3xy + x^2 \sin(x) = 0 )</td>
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<tr>
<td>(4) ( x \frac{d^4y}{dx^4} - 2 \frac{d^2y}{dx^2} + (1-x)y = 0 )</td>
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<tr>
<td>(5) ( y''' - 3y'' + 2y' - y = 2 )</td>
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<tr>
<td>(6) ( y''' - 3y'' + 2y' - xy = 0 )</td>
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<tr>
<td>(7) ( (y'')^3 + 2y' - e^y = 0 )</td>
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</table>
Differential Equations are used to describe almost every natural and manmade object that changes with time.

**Examples:**
1. Mechanics of rigid and flexible bodies
2. Behavior of water, light and electromechanical waves
3. Heat Transfer
4. Atomic behavior (Quantum Mechanics)
5. Fluid flow
6. Behavior of stocks, bonds, and options
7. Population growth
8. Radiation decay
9. Systems of masses and springs; electric circuits
10. Airplane and space flight
11. Economic and monetary changes
The idea is to find the unknown function $y$ that makes the differential equation true. Differential equations are classified in various ways.

The **Order** of a DE is the order of the highest derivative that appears in the equation.

An equation is **linear** if $y$ and each derivative that appear are raised to the first power only and multiplied at most by a known function of $x$. 
(1) \[ \frac{dy}{dx} = \cos(x) \]  
**First Order and linear**

(2) \[ \frac{dy}{dx} = y + x^2 \]

(3) \[ \frac{d^2 y}{dx^2} + e^x \frac{dy}{dx} + 3xy^3 + x^2 \sin(x) = 0 \]  
**Second Order and nonlinear**

(4) \[ x \frac{d^4 y}{dx^4} - 2 \frac{d^2 y}{dx^2} + (1-x)y = 0 \]  
**Order 4 and linear**

(5) \[ y''' - 3y'' + 2y' - y = 2 \]  
**Order 3 and linear**

(6) \[ y''' - 3y'' + 2y' - xy = 0 \]

(7) \[ (y'')^3 + 2y' - e^y = 0 \]  
**Order 3 and nonlinear**
In this course we will only study first order equations, both linear and non linear.

**Definition.** A *solution* of a differential equation is a function $f(x)$ that makes the equation true when it is substituted for $y$. The graph of a solution is called an *integral curve* for the DE.

**Example:** Find all solutions to the differential equation

$$\frac{dy}{dx} = x^2$$

And plot some integral curves.

**Solution.** By the usual methods of calculus, we see that

$$y = \int x^2 \, dx = \frac{x^3}{3} + C$$
Some integral curves are shown below. They are the graphs of the solution given previously for $C = 0, 1, 2, -3, -5$

Note that there are many solutions (integral curves), and to specify one particular one, you must have additional information.
**Example:** Find all solutions to the differential equation

\[ y' + 2y = 0 \]

Even for this simple generalization, integration does not produce the solutions. We are looking for a function whose derivative is a multiple of itself. This leads us to try a function of the form

\[ y = e^{ax} \]

Substituting this into the DE, we see that \[ y' = ae^{ax} \]
so the DE becomes:

\[ ae^{ax} + 2e^{ax} = (a + 2)e^{ax} = 0 \]

Since the exponential is never 0, we see that the proposed function can be a solution if and only if \( (a + 2) = 0 \) or \( a = -2 \). Thus one solution is

\[ y = e^{-2x} \]
By substitution, we can easily see that the functions \( y = Ce^{-2x} \) are solutions, for any \( C \). It can be shown that there are no other solutions to this DE. Some integral curves are shown below.
Again in the previous example, we saw that there are in general infinitely many solutions to a DE, and extra information is needed to narrow the solution to just one function. The extra information we will use comes in the form of an initial condition, specifying that we want a solution that has a specified value at some specified point, that is $y(x_0) = y_0$.

Thus the following are all initial conditions.

$$y(0)=0 \quad y(0)=-6 \quad y(1)=4 \quad y(-9)=6$$

An initial value problem is a first order differential equation together with an initial condition. If there is a solution to the DE that also satisfies the initial condition, it is called a solution to the initial value problem. There will be at most one such solution.
Example: Find the solution to the initial value problem

\[ \frac{dy}{dx} = x^2; \quad y(1) = 2 \]

We saw before that all solutions were of the form

\[ y = \frac{x^3}{3} + C \]

In order to satisfy the initial condition, we require that

\[ 2 = y(1) = \frac{1}{3} + C \]

so that \( C = \frac{5}{3} \), and the solution is

\[ y = \frac{x^3}{3} + \frac{5}{3} \]
We can also interpret the initial condition geometrically. To say that \( y(x_0) = y_0 \) is to say that we want the particular integral curve that goes through the point \((x_0, y_0)\). This is shown below for the previous example. The correct integral curve is in red.
Example: Find the solution to the initial value problem
\[ y' + 2y = 0; y(0) = 2 \]

We saw before that all solutions were of the form
\[ y = Ce^{-2x} \]
In order to satisfy the initial condition, we require that
\[ 2 = y(0) = Ce^{-2 \times 0} = C \]
so that \( C = 2 \), and the solution is \( y = 2e^{-2x} \)

Once again, this says that the integral curve corresponding to the solution we want must go through the point \((0, 2)\). This is shown in the following picture.
The solution to the initial value problem has its integral curve shown in red.
We can therefore view the solution to every initial value problem as a function that satisfies a first order differential equation and whose graph passes through a specified point \((x_0, y_0)\) in the plane.

The most general first order DE is of the form

\[ F(x, y, y') = 0 \quad (1) \]

Where \(F\) can be any reasonable formula with three variables. We will be interested in those equations which we can solve for \(y'\) and therefore which can be written as

\[ y' = f(x, y) \]
In general such DE cannot be solved explicitly. However, when we combine them with initial conditions, there are effective and interesting ways to approximate solutions numerically and graphically.

On the other hand, there are special types of first order differential equations that we can solve explicitly, and we will look at some of these types now.

1. We can use ordinary integration methods to solve first order DEs of the form

\[ \frac{dy}{dx} = f(x) \]

as we have seen.
2. If the DE can be expressed in the form

\[
\frac{dy}{dx} = g(x) \quad h(y)
\]

We say that the equation is separable. Essentially this means that we can separate the \(x\) and \(y\) parts, and combine each with its own differential. Thus we can write the equation as

\[
h(y)dy = g(x)dx
\]

In this case we can integrate the left and right sides separately, and the results must agree up to a constant. In other words, we have

\[
\int h(y)dy = \int g(x)dx \quad (2)
\]

If the resulting algebraic equation can be solved explicitly for \(y\), then these \(y\)’s will be the solutions of the DE. If not, we say that (2) defines the solution \textit{implicitly}. 

Example. Solve the differential equation
\[ \frac{dy}{dx} = x^3 y^2 \]

Then solve the initial value problem
\[ \frac{dy}{dx} = x^3 y^2; y(0) = 4 \]

Solution: This equation is separable, and by separating the variables, we get
\[ \frac{dy}{y^2} = x^3 dx \]
thus
\[ \int \frac{dy}{y^2} = \int x^3 dx \text{ or } -1 = \frac{x^4}{4} + C \]
so
\[ y = -\frac{1}{\frac{x^4}{4} + C} \]
To solve the initial value problem we need a solution that satisfies the initial condition. Thus we need

\[ 4 = y(0) = -\frac{1}{C} \]

or

\[ C = -\frac{1}{4} \]

Therefore the solution to the initial value problem is

\[ y = -\frac{1}{x^4 - 1} = -\frac{4}{1 - x^4} = \frac{4}{x^4 - 1} = \frac{4}{x^4 - 1} \]
Example. Solve the differential equation
\[
\frac{dy}{dx} = (1 + y^2)x^2
\]

Solution: This equation is separable, and by separating the variables, we get
\[
\frac{dy}{x^2dx} = \frac{1}{(1+y^2)}dx
\]
thus
\[
\int \frac{dy}{1+y^2} = \int x^2 dx \text{ or } \tan^{-1}(y) = \frac{x^3}{3} + C
\]
so
\[
y = \tan \left( \frac{x^3}{3} + C \right)
\]
Example. Solve the differential equation
\[ y' - (1+x)(1+y^2) = 0 \]

Solution: This equation is separable, and by separating the variables, we get
\[ \frac{dy}{dx} = (1+x)(1+y^2) \]
so
\[ \frac{dy}{1+y^2} = (1+x)dx \]
thus
\[ \int \frac{dy}{1+y^2} = \int (1+x)dx \]
or
\[ \tan^{-1}(y) = x + \frac{x^2}{2} + C \]
so
\[ y = \tan \left( x + \frac{x^2}{2} + C \right) \]
Example. Solve the differential equation

$$3\tan y \frac{dy}{dx} \sec x = 0$$

Solution: $$3\tan y \frac{dy}{dx} \sec x = 0$$ is equivalent with

$$\frac{dy}{dx} \sec x = 3\tan y$$ or $$\cot y dy = 3\cos x dx$$

thus

$$\ln |\sin y| = 3\sin x + C$$

so

$$|\sin y| = Ce^{3\sin x}$$

This is an implicit solution.
\[ |\sin y| = e^3 \sin x \]