1. Let \( \mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} \) and \( \mathbf{v} = \mathbf{i} + \mathbf{j} - 3\mathbf{k} \). Find: (4 points each)

(a) The cosine of the angle between \( \mathbf{u} \) and \( \mathbf{v} \).

Answer: \( \cos \theta = \frac{-2}{\sqrt{6} \sqrt{11}} \)

(b) A vector that is perpendicular to both \( \mathbf{u} \) and \( \mathbf{v} \).

Answer: \( \mathbf{u} \times \mathbf{v} = 2\mathbf{i} + 7\mathbf{j} + 3\mathbf{k} \)

2. (8 points) Determine parametric equations for the line which passes through the point \( P(1,1,2) \) and is perpendicular to the plane \( 2x - 3y + z = 15 \). Then find the coordinates of the point where the line intersects the plane.

Answer: Parametric equations are \( \begin{align*}
x &= 1 + 2t \\
y &= 1 - 3t \\
z &= 2 + t
\end{align*} \) and the point of intersection is \( (3,-2,3) \).
3. (8 pts) Given that the position vector of a moving particle is \( \mathbf{r}(t) = \left\langle \frac{t^3}{3}, \frac{t^2}{2}, t \right\rangle \), find the velocity vector, the acceleration vector, the speed \( \frac{ds}{dt} \), and the unit tangent vector when \( t = 1 \).

\[ \text{Answer: } \mathbf{v}(1) = \langle 1, 1, 1 \rangle, \quad \mathbf{a}(1) = \langle 2, 1, 0 \rangle, \quad |\mathbf{v}(1)| = \sqrt{3}, \quad \mathbf{T}(1) = \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}} \]

4. (8 pts) Describe the surface corresponding to the equation \( x^2 + y^2 + z^2 = 2z \), then write the equation in cylindrical coordinates and in spherical coordinates.

\[ \text{Answer: } \text{The surface is the sphere with radius 1 and center at } (0,0,1). \text{ The cylindrical equation is } r^2 + z^2 = 2z, \text{ and in spherical coordinates the equation is } \rho^2 = 2\rho \cos \phi. \]
5. (8 pts) Let \( f(x, y) = 3x^4 - xe^{-2y} \). Find \( \frac{\partial f}{\partial x} \), \( \frac{\partial f}{\partial y} \), \( \frac{\partial^2 f}{\partial x^2} \), and \( \frac{\partial^2 f}{\partial y \partial x} \).

Answer. \( \frac{\partial f}{\partial x} = 12x^3 - e^{-2y} \), \( \frac{\partial f}{\partial y} = 2xe^{-2y} \), \( \frac{\partial^2 f}{\partial x^2} = 36x^2 \), \( \frac{\partial^2 f}{\partial y \partial x} = 2e^{-2y} \).

6. (8 pts) The two legs \( x, y \) of a right triangle (see figure) vary with time in such a way that \( \frac{dx}{dt} = 0.1 \) in/s and \( \frac{dy}{dt} = -0.1 \) in/s. Find the rate of change of the hypotenuse at the instant when \( x = 3 \) in. and \( y = 4 \) in.

Answer. \( \frac{dh}{dt} = -0.02 \) when \( x = 3 \) and \( y = 4 \).
7. (8 pts) Suppose the temperature at a point \((x,y)\) in a flat plate be given by \(T(x,y) = 3x^2 + 2xy\). Find the rate of change of temperature at the point \((1,-2)\) in the direction of the vector \(v = 2i - 3j\). In what direction is the temperature increasing most rapidly at the point \((1,-2)\)?

\[
\nabla T(1,-2) \cdot \left( \frac{v}{|v|} \right) = \frac{-2}{\sqrt{13}} ; \text{ the temperature is increasing most rapidly in the direction of the vector } \nabla T(1,-2) = 2i + 2j
\n\]

8. (8 pts) A shipping container is to be constructed in the form of an open-top rectangular box. To accommodate the weight of the contents, the bottom of the box is to be made of materials costing three times as much as that used for the sides. The volume of the box is to be 1500 ft\(^3\). Find the dimensions that should be chosen in order to minimize the cost of the materials used.

\textbf{Answer.} The bottom should be a 10 ft. by 10 ft. square, and the height should be 15 ft.
9. (10 pts) Evaluate the integral \[ \iint_R xy \, dA \] where \( R \) is the triangular region in the first quadrant bounded by \( 2x + y = 2 \), \( y = 0 \), and \( x = 0 \).

**Answer.** \( \iint_R xy \, dA = \frac{1}{6} \)

10. (8 pts) Sketch the region of integration corresponding to \( \int_0^2 \int_0^{4-x^2} f(x,y) \, dy \, dx \), then express as an equivalent integral with the order of integration reversed.

**Answer.**
\[
\int_0^2 \int_0^{4-x^2} f(x,y) \, dy \, dx = \int_0^4 \int_0^{\sqrt{4-y}} f(x,y) \, dx \, dy
\]
11. (10 pts) Use polar coordinates to evaluate \( \iint_{R} y \, dA \) where \( R \) is the region in the first quadrant bounded by the circle \( x^2 + y^2 = 4 \) and the lines \( y = x \) and \( y = 0 \) (x-axis).

\[
\text{Answer. } \int \int_{R} y \, dA = \int_{0}^{\pi/4} \int_{0}^{2} r \sin \theta \, r \, dr \, d\theta = \frac{8}{3\sqrt{2}}
\]

12. (8 pts) Set up an integral that represents the volume of the solid region that is under the paraboloid \( z = 4 - x^2 - y^2 \) and above the xy-plane. DO NOT EVALUATE THE INTEGRAL.

[Note that you are allowed considerable flexibility as to how to go about this problem. In particular, you may set it up as either a double integral or a triple integral, and you may use either rectangular or polar/cylindrical) coordinates, etc.]

\[
\text{Answer: } V = \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{4-r^2} dz \, r \, dr \, d\theta = \int_{0}^{2\pi} \int_{0}^{2} (4-r^2) \, r \, dr \, d\theta
\]
**MISCELLANEOUS FORMULAS**

\[ \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}| |\mathbf{b}| \cos(\theta) \]

\[ \text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \]

\[ \text{proj}_{\mathbf{a}} \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} \]

\[ \mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \]

\[ |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin(\theta) \]

\[ \mathbf{L} : \begin{align*} x &= x_0 + at \\
y &= y_0 + bt \\
z &= z_0 + ct \end{align*} \]

\[ \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \]

\[ \mathbf{a}(x - x_0) + \mathbf{b}(y - y_0) + \mathbf{c}(z - z_0) = 0 \quad \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} = 0 \]

\[ \frac{dr}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \]

\[ \frac{ds}{dt} = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 + \left( \frac{dz}{dt} \right)^2} \]

\[ x = r \cos \theta \quad y = r \sin \theta \quad z = z \quad dV = dz \, r \, dr \, d\theta \]

\[ x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi \quad dV = \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi \]

\[ z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) \]

\[ dz = \frac{\partial f}{\partial x}(x_0, y_0)dx + \frac{\partial f}{\partial y}(x_0, y_0)dy \]

\[ \nabla f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) \mathbf{i} + \frac{\partial f}{\partial y}(x_0, y_0) \mathbf{j} \]

\[ D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} \]

\[ \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \]

\[ \frac{dz}{ds} = \frac{\partial z}{\partial x} \frac{dx}{ds} + \frac{\partial z}{\partial y} \frac{dy}{ds} \]