1. (6 pts) Show that the equation $x^2 + y^2 + z^2 = 2x$ represents a sphere, and find its center and radius.

**Solution.** Upon completing the square on the variable $x$, we have:

$$(x^2 - 2x + 1) + y^2 + z^2 = 0 + 1$$

$$(x - 1)^2 + y^2 + z^2 = 1$$

Thus the equation represents a sphere with center at the point $(1, 0, 0)$ and radius 1.

2. Let $u = 3i - j - 2k$ and $v = 2i - 3j + k$. Find: (5 pts each)

(a) The dot product of $u$ and $v$.

**Solution.**

$$u \cdot v = (3)(2) + (-1)(-3) + (-2)(1) = 6 + 3 - 2 = 7$$

(b) The angle between $u$ and $v$.

**Solution.**

$$\cos \theta = \frac{u \cdot v}{\|u\| \|v\|} = \frac{7}{\sqrt{3^2 + (-1)^2 + (-2)^2} \sqrt{2^2 + (-3)^2 + 1^2}} = \frac{7}{\sqrt{14} \sqrt{14}} = \frac{1}{2}$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \quad \text{(or 60°)}$$

(c) The scalar projection of $u$ onto $v$.

**Solution.**

$$\text{comp}_v u = \frac{u \cdot v}{\|v\|} = \frac{7}{\sqrt{14}}$$

(d) The vector projection of $u$ onto $v$.

**Solution.**

$$\text{proj}_v u = \left(\frac{u \cdot v}{\|v\|^2}\right) \frac{v}{\|v\|} = \left(\frac{7}{\sqrt{14} \sqrt{14}}\right) \frac{2i - 3j + k}{2} = \frac{2i - 3j + k}{2} = i - \frac{3}{2}j + \frac{1}{2}k$$
3. Let \( \mathbf{u} = \langle 2, -1, 3 \rangle \) and \( \mathbf{v} = \langle 1, 1, -2 \rangle \). Find: (4 pts each)

(a) The cross product of \( \mathbf{u} \) and \( \mathbf{v} \).

Solution.

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
i & j & k \\
2 & -1 & 3 \\
1 & 1 & -2
\end{vmatrix} = (2 - 3)i - (4 - 3)j + (2 + 1)k = -i + 7j + 3k
\]

(b) A unit vector (length 1) which is perpendicular to \( \mathbf{u} \) and \( \mathbf{v} \).

Solution.

\[
\pm \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \pm \frac{-i + 7j + 3k}{\sqrt{(-1)^2 + (7)^2 + (3)^2}} = \pm \frac{-i + 7j + 3k}{\sqrt{59}}
\]

(c) The area of the parallelogram having \( \mathbf{u} \) and \( \mathbf{v} \) as adjacent edges.

Solution.

\[
|\mathbf{u} \times \mathbf{v}| = \sqrt{(-1)^2 + (7)^2 + (3)^2} = \sqrt{59}
\]

4. (12 pts) Find an equation for the plane that passes through the points \( P(1, -1, 2) \), \( Q(2, 3, -1) \) and \( R(-2, 0, 5) \).

Solution. The vector \( \mathbf{n} = \mathbf{PQ} \times \mathbf{PR} \) is perpendicular to the plane through \( P, Q, \) and \( R \).

\[
\mathbf{PQ} = \langle 2 - 1, 3 + 1, -1 - 2 \rangle = \langle 1, 4, -3 \rangle \quad \mathbf{PR} = \langle -2 - 1, 0 + 1, 5 - 2 \rangle = \langle -3, 1, 3 \rangle
\]

\[
\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix}
i & j & k \\
1 & 4 & -3 \\
-3 & 1 & 3
\end{vmatrix} = (12 + 3)i - (3 - 9)j + (1 + 12)k = 15i + 6j + 13k
\]

Using \( P(1, -1, 2) \) as base point, and \( \mathbf{n} = 15i + 6j + 13k \) as normal direction, an equation for the plane through \( P, Q, \) and \( R \) is:

\[
15(x - 1) + 6(y + 1) + 13(z - 2) = 0
\]

\[
15x + 6y + 13z = 35
\]
5. (12 pts) Find parametric equations for the line which passes through the point P(–2,0,1) and is parallel to the line \( \frac{x}{4} = \frac{y + 1}{-3} = \frac{z - 2}{5} \). Then find the point where the line intersects the xy–plane.

**Solution.** The vector \( \mathbf{a} = \langle 4, -3, 5 \rangle \) is parallel to the line that we seek. Using this direction, and P(–2,0,1) as base point, parametric equations for the line are:

\[
\begin{align*}
x &= -2 + 4t \\
y &= 0 - 3t \\
z &= 1 + 5t
\end{align*}
\]

The point where this line intersects the xy–plane corresponds to \( z = 1 + 5t = 0 \), i.e. \( t = -\frac{1}{5} \). Thus the point of intersection with the xy–plane is:

\[
\left( -2 - \frac{4}{5}, 0 + \frac{3}{5}, 1 - \frac{5}{5} \right) = \left( -\frac{14}{5}, \frac{3}{5}, 0 \right)
\]

6. (6 pts each) Use the axes provided to sketch each of the following surfaces. In each case, identify (by name) the type of surface.

(a) \( x^2 + y^2 + 4z^2 = 4 \)

**Solution.** The surface is the ellipsoid

\[
\frac{x^2}{4} + \frac{y^2}{4} + z^2 = 1
\]

with vertices at \((\pm 2, 0, 0)\), \((0, \pm 2, 0)\), and \((0, 0, \pm 1)\).

(b) \( x + y = 1 \)

**Solution.** The surface is a plane which is perpendicular to the xy–plane. The x–intercept is \((1,0,0)\), and the y–intercept is \((0,1,0)\).
7. Let \( C \) be the space curve corresponding to the vector valued function \( \mathbf{r}(t) = \langle \cos(t), 2t, \sin(t) \rangle \).

(a) (6 pts) Describe the curve \( C \) (identify it by name and plot it in some reasonable fashion).

**Solution.**

The curve is a helix (spiral) on the surface of the cylinder \( x^2 + z^2 = 1 \).

(b) (6 pts) Find a unit vector which is tangent to the curve at the point corresponding to \( t = \pi \).

**Solution.**

\[
\frac{d\mathbf{r}}{dt} = \langle -\sin(t), 2, \cos(t) \rangle \quad \frac{ds}{dt} = \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{(-\sin(t))^2 + (2)^2 + (\cos(t))^2} = \sqrt{5}
\]

\[
\mathbf{T}(t) = \frac{d\mathbf{r}}{ds} = \frac{\langle -\sin(t), 2, \cos(t) \rangle}{\sqrt{5}}
\]

\[
\mathbf{T}(\pi) = \frac{\langle 0, 2, -1 \rangle}{\sqrt{5}}
\]

(c) (6 pts) Find the arc length of the segment of the curve corresponding to \( 0 \leq t \leq \pi \).

**Solution.**

\[
L = \int_0^\pi \left( \frac{ds}{dt} \right) dt = \int_0^\pi \sqrt{5} \ dt = \sqrt{5} \pi \approx 7.02
\]

8. (8 pts) The rectangular coordinates of a point are \( (1,1,\sqrt{2}) \). Find the cylindrical coordinates of the point and the spherical coordinates of the point.

**Solution.**

Cylindrical: \( r = \sqrt{(1)^2 + (1)^2} = \sqrt{2} \), \( \theta = \frac{\pi}{4} \), \( z = \sqrt{2} \)

\[
\left( \sqrt{2}, \frac{\pi}{4}, \sqrt{2} \right)
\]

Spherical: \( \rho = \sqrt{(1)^2 + (1)^2 + (\sqrt{2})^2} = \sqrt{4} = 2 \), \( \theta = \frac{\pi}{4} \), \( \phi = \frac{\pi}{4} \)

\[
\left( 2, \frac{\pi}{4}, \frac{\pi}{4} \right)
\]