1. X is a metric space with metric d., and Y ⊆ X. Prove that Y is closed if and only if Y’ is open.

P. (a) Suppose Y is closed and x ∈ Y’. Since Y contains all of its cluster points, x cannot be a cluster point of Y. Thus some sphere $S_{\varepsilon}(x)$ contains no points from Y. This means that $S_{\varepsilon}(x) \subseteq Y’$. Thus every point of Y’ is an interior point, and so Y’ is open.

(b) Suppose that Y’ is open and that x ∈ Y’. Then some sphere $S_{\varepsilon}(x) \subseteq Y’$, so this sphere does not intersect Y. Hence x is not a cluster point of Y. This means that all cluster points of Y must belong to Y, and so Y is closed.

2. Show that the intersection of a finite number of open sets is open, and the union of any number of open sets is open.

(a) Let $O_1, O_2, \ldots, O_n$ be open sets, and let $x \in \bigcap_{k=1}^{n} O_k$. Then for each $k$ there is a sphere $S_k$ with center x and radius $\varepsilon_k$ that lies entirely in $O_k$. If $\varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_n)$, then $S_{\varepsilon} \subseteq \bigcap_{k=1}^{n} S_k \subseteq \bigcap_{k=1}^{n} O_k$, and so $x$ is an interior point of $\bigcap_{k=1}^{n} O_k$. Since x is arbitrary, $\bigcap_{k=1}^{n} O_k$ is an open set.

(b) Now let $T$ be any collection of open sets, and let $x \in \bigcup T$. Then we have $x \in O$, for some $O \in T$. For some $\varepsilon > 0$, we must then have $S_{\varepsilon}(x) \subseteq O \subseteq \bigcup T$, and so $x$ is a interior point of $\bigcup T$. Since $x$ is arbitrary, the set $\bigcup T$ is open.