1. If \( s = (x_n)_{1 \leq n < \infty} \) is a convergent sequence in a metric space \((X, d)\), then \( s \) is a Cauchy sequence.

P. Let \( \varepsilon > 0 \) be given. Since \( s \) converges to an element \( x \in X \), there is an integer \( N \) so that if \( n \geq N \), then \( \| x - x_n \| < \frac{\varepsilon}{2} \). Then if both \( n \) and \( m \) are \( \geq N \), we have \( \| x_n - x_m \| \leq \| x - x_n \| + \| x - x_m \| < \varepsilon \). Thus \( s \) is Cauchy.

2. If a subset \( Y \) of a metric space \((X, d)\) is totally bounded, then \( Y \) is bounded (entirely contained in some sphere).

P. Since \( Y \) is totally bounded, there exist points \( x_1, x_2, \ldots, x_n \) so that
\[
Y \subseteq \bigcup_{k=1}^{n} S_1(x_k).
\]
Let \( \gamma \) be the maximum of the distances \( d(x_1, x_k) \) for \( 2 \leq k \leq n \), and let \( \eta = 1 + \gamma \). We claim that \( Y \subseteq S_\eta(x_1) \). In fact if \( y \in Y \), then \( d(y, x_i) < 1 \) for some \( i \) between 1 and \( n \). Then \( d(y, x_1) \leq d(y, x_i) + d(x_i, x_1) \leq 1 + \gamma = \eta \).

3. Prove that if \( s = (x_n)_{1 \leq n < \infty} \) is a sequence in a metric space \((X, d)\) then
   (a) If \( s \) converges then \( s \) has exactly one limit point.
   (b) If \( X \) is compact and \( s \) has exactly one limit point, then \( s \) converges.

P. (a) Suppose that \( s \) converges to \( x \). The \( x \) is clearly a limit point of \( s \).
Now suppose that \( y \) is any other point of \( X \), and that \( \eta = d(x, y) > 0 \). Then there exists an \( N \) so that for \( n \geq N \), \( d(x, x_n) < \frac{\eta}{3} \). Thus the sphere \( S_{\frac{\eta}{3}}(y) \) can contain no term \( x_n \) for \( n \geq N \), since if it did contain some such point \( x_n \), then \( d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{2\eta}{3} \) which would be a contradiction. This means that \( y \) is not a limit point of \( s \), and so \( s \) has exactly one limit point.

(b) Suppose that \( X \) is compact and that \( s \) has exactly one limit point \( x \). If \( s \) does not converge to \( x \), then there is an \( \varepsilon > 0 \) so that \( d(x_n, x) \geq \varepsilon \) infinitely often, say for the subsequence \( t = (x_{n_k})_{1 \leq k < \infty} \). Since \( X \) is compact, \( t \) has a limit point \( y \). Then \( y \) is a limit point of \( s \), but \( d(x, y) = \lim_{k \to \infty} d(x, x_{n_k}) \geq \varepsilon \), and so \( y \) is not equal to \( x \). Thus \( s \) has two limit points, which is a contradiction. This means that the sequence \( s \) must converge to \( x \).