1. Show that the sequence
\[ f_n(x) = \frac{1}{1 + nx}, \quad 0 \leq x \leq 1 \] converges pointwise but not uniformly.

P. Clearly \( f_n(0) = 1 \) for all \( n \), so \( \lim_{n \to \infty} f_n(0) = 1 \). If \( x > 0 \), then \( nx \) tends to infinity, and therefore \( \lim_{n \to \infty} f_n(x) = 0 \). Thus the sequence converges pointwise. The convergence cannot be uniform since the limit function is
\[ f(x) = \begin{cases} 1 & |x| = 0 \\ 0 & |x| \neq 0 \end{cases} \] and this is not continuous.

For a direct proof, note that \( f_n \left( \frac{1}{n} \right) = \frac{1}{2} \), and so \( \|f - f_n\|_{\max} \geq 0.5 \). Thus the sequence cannot converge uniformly.

2. Let \( A \subseteq \mathbb{R} \) (the real numbers), \( A \neq \emptyset \), and define
\[ f_A(x) = \inf \{|x - a| : a \in A\}. \] Show that \( f_A \) is continuous on \( \mathbb{R} \).

Let \( x \) and \( y \) be fixed real numbers, and \( \varepsilon > 0 \). By definition there must be an element \( a \) in \( A \) so that \( |y - a| < f_A(y) + \varepsilon \).
Then \( |x - a| \leq |x - y| + |y - a| < |x - y| + f_A(y) + \varepsilon \). This means that \( f_A(x) \leq |x - y| + f_A(y) + \varepsilon \). Since this is true for all \( \varepsilon \), we must have \( f_A(x) \leq |x - y| + f_A(y) \), or \( f_A(x) - f_A(y) \leq |x - y| \).

By interchanging the role of \( x \) and \( y \), we see that
\[ |f_A(x) - f_A(y)| \leq |x - y|, \] and so \( f_A \) is continuous (we can take \( \delta = \varepsilon \)).

3. Let \( A \) and \( B \) be subsets of \( \mathbb{R} \) and define
\[ d(A,B) = \inf \{d(a,b) : a \in A, b \in B\} \]. Prove that if \( A \) and \( B \) are compact and \( d(A,B) = 0 \), then \( A \cap B \neq \emptyset \).

P. For each \( n \) we can find points \( a_n \in A \) and \( b_n \in B \) so that \( d(a_n, b_n) < \frac{1}{n} \).
Since \( A \) and \( B \) are compact, the sequences \( s = (a_n)_{1 \leq n < \infty} \) and \( t = (b_n)_{1 \leq n < \infty} \)
must have convergent subsequences. Thus (replacing s and t with subsequences if necessary) we may assume that s and t converge to a and b respectively. Since A and B are closed (being compact), we have $a \in A$ and $b \in B$. Then $d(a,b) = \lim_{n \to \infty} d(a_n,b_n) = 0$. This means that $a = b$, and so $A \cap B \neq \emptyset$. 