Test 2 Study Guide for Certain types of Problems.

1. Given a set of vectors \( \{v_1, v_2, \ldots, v_k\} \) in \( \mathbb{R}^n \), determine if this is a linearly independent or linearly dependent set.

**Solution.** Make a matrix \( A \) using the vectors as columns. Form the system of equations \( Ax = x_1v_1 + x_2v_2 + \ldots + x_kv_k = 0 \). Put \( [A : 0] \) in reduced row echelon form and determine if there is only the trivial solution to the system (in which case the \( v \)'s are independent) or infinitely many solutions (in which case the \( v \)'s are dependent).

2. Is a given vector \( w \) in \( \mathbb{R}^n \) a linear combination of a set of vectors \( \{v_1, v_2, \ldots, v_k\} \), or not?

**Solution.** Make a matrix \( A \) using the vectors as columns. Form the system of equations \( Ax = w \). Put \( [A : w] \) in reduced row echelon form and determine if there are any solutions (in which case \( w \) is a linear combination) or not (in which case \( w \) is not a linear combination). To write \( w \) as a specific linear combination of the \( v \)'s, choose any desired solution to the system \( Ax = 0 \).

3. Does a given set \( \{v_1, v_2, \ldots, v_k\} \) of vectors span \( \mathbb{R}^n \)? (Here we must have \( k \geq n \), or the answer is definitely no).

**Solution.** Make a matrix \( A \) using the vectors as columns. Form the system of equations \( Ax = w \), where \( w \) is an arbitrary vector in \( \mathbb{R}^n \) (that is \( w \) is formed using parameters, i.e. letters). Put \( [A : w] \) in reduced row echelon form and determine if there are solutions no matter what values are given to the parameters (in which case the set does span \( \mathbb{R}^n \)) or whether some values of the parameters produce a system with no solutions (in which case the set does not span \( \mathbb{R}^n \)).

4. Let \( A \) be a matrix. Find bases for, and the dimension of, the subspaces \( \text{Row}(A) \), \( \text{Col}(A) \), and \( \text{Nul}(A) \).

**Solution.** Put \( A \) in reduced row echelon form \( R \). The nonzero rows of \( R \) form a basis for \( \text{Row}(A) \), and so the dimension of \( \text{Row}(A) \) is the number of nonzero rows of \( R \), which is the number of leading 1’s.

To find a basis for \( \text{Col}(A) \), note the columns of \( R \) that contain leading 1’s. The corresponding columns of \( A \) form a basis for \( \text{Col}(A) \). Thus the dimension of \( \text{Col}(A) \) is also the number of leading 1’s, and so equals the dimension of \( \text{Row}(A) \). Be careful –
the columns of $R$ containing leading 1’s are not in general a basis for $Col(A)$ (or even in that space).

To find a basis for $Nul(A)$, write down the parametric equations for the solutions by inspection from $R$, as usual. Then set each free parameter in turn to 1 and all others to 0. The resulting vectors form a basis for $Nul(A)$, and so the dimension of $Nul(A)$ is equal to the number of free parameters.

5. Find a basis for, and the dimension of the subspace $W$ of $R^n$ spanned by the vectors $\{v_1, v_2, \ldots, v_k\}$.

**Solution1.** Make a matrix $A$ using the vectors as rows. Then $W = Row(A)$, and you proceed as in 4. to find a basis. This provides a simple basis for $W$ that generally consists of vectors other than the $v$’s.

**Solution2.** Make a matrix $A$ using the vectors as columns. Then $W = Col(A)$, and you proceed as in 4. to find a basis. This basis comes from the columns of $A$, that is it is a subset of the original set $\{v_1, v_2, \ldots, v_k\}$.

6. Find a basis for the subspace $\{v_1, v_2, \ldots, v_k\}^\perp$ consisting of all vectors orthogonal to the set $\{v_1, v_2, \ldots, v_k\}$.

**Solution.** Make a matrix $A$ using the vectors as rows. By the nature of matrix multiplication, we see that $Ax = 0$ if and only if $x$ is orthogonal to all rows of $A$, that is if and only if $x$ is in $\{v_1, v_2, \ldots, v_k\}^\perp$. Thus $\{v_1, v_2, \ldots, v_k\}^\perp = Nul(A)$, and you proceed as in 4. to find a basis.