1. Let \( u = (1, 2, -1, 0, a) \) and \( v = (0, 2, 0, 3, -1) \) be two vectors in \( \mathbb{R}^5 \).
   a. (6 pts.) For what value of \( a \) are the vectors \( u \) and \( v \) orthogonal (perpendicular)?

   **Solution.**
   \[
   u \cdot v = 0 + 4 + 0 + 0 - a = 4 - a.
   \]
   Thus we need \( a = 4 \), for perpendicularity.

   b. (6 pts) If \( a = 1 \), what is the cosine of the angle between \( u \) and \( v \)?

   **Solution.**
   \[
   \|u\| = \sqrt{u \cdot u} = \sqrt{1+4+1+0+1} = \sqrt{7} \]
   \[
   \|v\| = \sqrt{v \cdot v} = \sqrt{0+4+0+9+1} = \sqrt{14} \]
   \[
   \cos(\theta) = \frac{u \cdot v}{\|u\| \|v\|} = \frac{3}{\sqrt{98}}
   \]

   c. (2 pts) How would you describe geometrically the set of all linear combinations of the vectors \( u \) and \( v \) in part b?

   **Solution.**
   A plane through the origin in \( \mathbb{R}^5 \)

2. (3 pts each) Let \( A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 2 & 1 \end{bmatrix} \), \( B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 1 \end{bmatrix} \). Compute the following matrices, or state that they cannot be computed.

   a. \( AB = \begin{bmatrix} 3 & 5 \\ -11 & 2 \end{bmatrix} \)
   b. \( BA = \begin{bmatrix} 0 & 8 & 1 \\ -6 & 4 & 2 \\ -5 & -2 & 1 \end{bmatrix} \)
   c. \( A + B \) cannot be computed.

   d. \( A+B^T = \begin{bmatrix} 4 & 2 & -2 \\ -2 & 4 & 2 \end{bmatrix} \)
3. a. (10 pts) Solve the following system of linear equations:

\[
\begin{align*}
    x_1 + 2x_2 + 4x_3 + 5x_4 &= 4 \\
    x_2 + 2x_3 + 2x_4 &= 1
\end{align*}
\]

**Solution.** The augmented matrix is

\[
\begin{bmatrix}
    1 & 2 & 4 & 5 & 4 \\
    0 & 1 & 2 & 2 & 1
\end{bmatrix} = \begin{bmatrix}
    1 & 0 & 0 & 1 & 2 \\
    0 & 1 & 2 & 2 & 1
\end{bmatrix}.
\]

Thus the parametric form of the solution is

\[
\begin{align*}
    x_1 &= 2 - s \\
    x_2 &= 1 - 2r - 2s \\
    x_3 &= r \\
    x_4 &= s
\end{align*}
\]

b. (10 pts) Suppose that the matrix

\[
\begin{bmatrix}
    2 & 1 & 2 & 2 & 1 \\
    1 & 3 & 4 & 2 & \phantom{0}
\end{bmatrix}
\]

is the augmented matrix of a system of equations. Find conditions on the constant \( a \) under which the system has – no solutions, 1 solution, or infinitely many solutions.

**Solution.** There are three cases.

Case 1. \( a = 2 \). Then the second matrix becomes

\[
\begin{bmatrix}
    1 & 2 & 2 & 1 \\
    0 & 1 & 2 & 1
\end{bmatrix},
\]

and so there are infinitely many solutions.

Case 2. \( a = -2 \). Then the second matrix becomes

\[
\begin{bmatrix}
    1 & 2 & 2 & 1 \\
    0 & 1 & 2 & 1
\end{bmatrix},
\]

and so there are no solutions.

Case 3. \( a^2 \neq 4 \). Then the second matrix becomes

\[
\begin{bmatrix}
    1 & 2 & 2 & 1 \\
    0 & 1 & 2 & 1
\end{bmatrix},
\]

and so the solution is unique.
4. (15 pts) Find the inverse of the matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
1 & 3 & 7
\end{pmatrix}
\]

Solution.
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
1 & 3 & 7
\end{pmatrix} \sim
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 2 \\
1 & 3 & 7
\end{pmatrix} \sim
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 3 & 7
\end{pmatrix} \sim
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 0 & -1
\end{pmatrix}
\]

Thus the inverse is
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 2 \\
0 & 3 & -7
\end{pmatrix}
\]

5. a. (12 pts) Compute the determinant of the matrix
\[
\begin{pmatrix}
7 & 2 & 9 \\
1 & 0 & 3 \\
0 & 0 & 1
\end{pmatrix}
\]

Solution.
\[
\begin{vmatrix}
7 & 2 & 9 \\
1 & 0 & 3 \\
0 & 0 & 1
\end{vmatrix} = 7 \begin{vmatrix}
1 & 0 \\
0 & 1
\end{vmatrix} + 2 \begin{vmatrix}
1 & 3 \\
0 & 1
\end{vmatrix} + 9 \begin{vmatrix}
1 & 0 \\
0 & 3
\end{vmatrix} = 2 - 3 = -1
\]

b. (12 pts) Suppose that \(A\) and \(B\) are 3 by 3 matrices, and that \(\det(A) = 2, \ \det(B) = -1\). Find the determinant of the matrix \(3A^2B^{-1}\).

Solution. \(\det\left(3A^2B^{-1}\right) = 3^3 \frac{\det(A)^2}{\det(B)} = 27(-4) = -108\).
6. (3 pts each) Mark each of the following statements as either true (T) or false (F).
   a. A system of 2 equations in 4 unknowns cannot have a unique solution  
      ________
   b. A system of 4 equations in 2 unknowns is inconsistent.  
      ________
   c. If det(A)=3 and det(B) = 5, then det (A + B) = 8  
      ________
   d. A homogeneous system of equations (constants all 0) must be consistent  
      ________
   e. The set of all linear combinations of two vectors in \( \mathbb{R}^3 \) always forms a plane.  
      ________

Solution.  (a) T  (b) F  (c)  F  (d)  T  (e)  F

7. (Extra Credit 10 pts) Suppose that a linear system of equations can be written as
   \[
   A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and that } A^{-1} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 1 & 1 \end{bmatrix}. \text{ Find the solution vector } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.
   \]

Solution. If \( Ax = b \), then \( x = A^{-1}b \). Thus
   \[
   \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 5 \end{bmatrix}.
   \]

Similarly if \( b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \), then \( x = \begin{bmatrix} 2 \\ -3 \\ -2 \\ 3 \end{bmatrix} \).