ORTHOGONAL EXPANSIONS & PROJECTIONS
(Summary)

Definition. A set of vectors \( S = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) in \( \mathbb{R}^n \) is called an **orthogonal set** if each pair of vectors from the set is orthogonal, i.e. if \( \mathbf{u}_i \perp \mathbf{u}_j \) \( (\mathbf{u}_i \cdot \mathbf{u}_j = 0) \) for \( i \neq j \).

**Theorem.** If \( S = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) is an orthogonal set of nonzero vectors in \( \mathbb{R}^n \) \( (1 < p \leq n) \), then \( S \) is a linearly independent set and hence is a basis for the subspace \( W = \text{Span}(S) \).

**Theorem.** If \( S = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) is an orthogonal basis for a subspace \( W \) of \( \mathbb{R}^n \), then each vector \( \mathbf{y} \) in \( W \) can be expressed in terms of the basis vectors as follows:

\[
\mathbf{y} = (\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}) \mathbf{u}_1 + (\frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}) \mathbf{u}_2 + \ldots + (\frac{\mathbf{y} \cdot \mathbf{u}_p}{\mathbf{u}_p \cdot \mathbf{u}_p}) \mathbf{u}_p
\]

Definition. A set of vectors \( \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) in \( \mathbb{R}^n \) is called an **orthonormal set** if each pair of vectors from the set is orthogonal \( (\mathbf{u}_i \cdot \mathbf{u}_j = 0 \text{ for } i \neq j) \) and each vector in the set has unit length \( (\text{i.e. } \mathbf{u}_j \cdot \mathbf{u}_j = \| \mathbf{u}_j \|^2 = 1 \text{ for all } j) \). In Kronecker Delta Notation:

\[
\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

**Theorem.** If \( S = \{ \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p \} \) is an orthonormal set, then \( S \) is a basis (called an **orthonormal basis** (ON basis)) for the subspace \( W = \text{Span}(S) \), and each vector \( \mathbf{y} \) in \( W \) can be expressed in terms of the basis vectors as follows:

\[
\mathbf{y} = (\mathbf{y} \cdot \mathbf{u}_1) \mathbf{u}_1 + (\mathbf{y} \cdot \mathbf{u}_2) \mathbf{u}_2 + \ldots + (\mathbf{y} \cdot \mathbf{u}_p) \mathbf{u}_p
\]
Theorem. If \( \hat{u}, \hat{y} \in \mathbb{R}^n \) (with \( \hat{u} \neq \hat{0} \)) then there is a scalar \( \alpha \) and a vector \( \hat{z} \) such that \( \hat{y} = \alpha \hat{u} + \hat{z} \) and \( \hat{z} = \hat{y} - \alpha \hat{u} \) is orthogonal to \( \hat{u} \):

\[
\alpha = \frac{\hat{y} \cdot \hat{u}}{\hat{u} \cdot \hat{u}}
\]

\[
\alpha \hat{u} = (\frac{\hat{y} \cdot \hat{u}}{\hat{u} \cdot \hat{u}}) \hat{u} = \text{orthogonal projection of } \hat{y} \text{ onto } \hat{u}
\]

\[
\hat{z} = \hat{y} - (\frac{\hat{y} \cdot \hat{u}}{\hat{u} \cdot \hat{u}}) \hat{u} = \text{component of } \hat{y} \text{ orthogonal to } \hat{u}
\]

Theorem. If \( S \) is any subset of \( \mathbb{R}^n \), then the set

\[
S^\perp = \{ \hat{z} \in \mathbb{R}^n \mid \hat{z} \cdot \hat{s} = 0 \text{ for all } \hat{s} \in S \}
\]

is a subspace of \( \mathbb{R}^n \) (called the orthogonal complement of \( S \)).

Theorem. Let \( W \) be a subspace of \( \mathbb{R}^n \). Then each \( \hat{y} \in \mathbb{R}^n \) can be written (uniquely) in the form

\[
\hat{y} = \hat{w} + \hat{z}
\]

where \( \hat{w} \in W \) and \( \hat{z} \in W^\perp \). The vector \( \hat{w} \) (commonly denoted by \( \text{proj}_W \hat{y} \)) is referred as the orthogonal projection of \( \hat{y} \) onto \( W \). If \( \{ \hat{u}_1, \hat{u}_2, \ldots, \hat{u}_p \} \) is an orthogonal basis for \( W \), then

\[
\text{proj}_W \hat{y} = (\hat{y} \cdot \hat{u}_1) \hat{u}_1 + (\hat{y} \cdot \hat{u}_2) \hat{u}_2 + \ldots + (\hat{y} \cdot \hat{u}_p) \hat{u}_p
\]

If, in addition, \( \hat{u}_j \cdot \hat{u}_j = ||\hat{u}_j||^2 = 1 \) for each \( j \) (i.e. \( \{ \hat{u}_1, \hat{u}_2, \ldots, \hat{u}_p \} \) is an ON basis for \( W \)) then

\[
\text{proj}_W \hat{y} = (\hat{y} \cdot \hat{u}_1) \hat{u}_1 + (\hat{y} \cdot \hat{u}_2) \hat{u}_2 + \ldots + (\hat{y} \cdot \hat{u}_p) \hat{u}_p
\]