The following list of problems represents a spectrum of the topics that were covered throughout the quarter.

Many problems are adopted from our quizzes and midterm.

The final exam will be a mild variant of a subset of the union of this mock final, quizzes and midterm.

Do not ignore this mock final, because you are not going to be misled.

(Formation of Matrix Equations)
1. Use Kirchhoff’s Law to write a matrix equation that determines the loop currents $I_1, I_2, I_3$ and $I_4$ in the circuit of problem 6 on p.93 of the text. You don’t need to solve the system.
2. Write an equivalent matrix equation for the following system. You don’t need to solve the system.
   \[
   \begin{align*}
   x_1 + 2x_2 - x_3 - x_4 - 8x_5 &= 9 \\
   x_2 + x_2 - 4x_3 - x_4 &= 6 \\
   x_2 + 3x_3 + x_5 &= 6
   \end{align*}
   \]
3. (a) Write an equivalent vector equation for the system in problem (2).
   (b) Write a system of equations that is equivalent to the vector equation below. You don’t need to solve the system.
   \[
   \begin{bmatrix}
   3 \\
   1 \\
   -1 \\
   0
   \end{bmatrix}
   +
   \begin{bmatrix}
   0 \\
   2 \\
   3 \\
   -1
   \end{bmatrix}
   +
   \begin{bmatrix}
   5 \\
   2 \\
   3 \\
   -2
   \end{bmatrix}
   =
   \begin{bmatrix}
   2 \\
   1 \\
   8 \\
   3
   \end{bmatrix}
   \]
   (c) Write an equivalent matrix equation for the vector equation in (b).

(Solving a Linear System)
1. Solve the linear system for the unknown $y$. You don’t need to solve for the other unknowns. You may use the Cramer’s rule.
   \[
   \begin{align*}
   2x + 3y - 4z - u &= 1 \\
   y - 2z + 2u &= 0 \\
   z + 3u &= 0 \\
   -z + u &= 1
   \end{align*}
   \]
2. Reduce the augmented matrix of the system to its reduced row echelon form and find the general solution of the system
\[
\begin{align*}
\begin{aligned}
x_1 + x_2 - x_3 - x_4 - 8x_5 &= 9 \\
x_2 + 4x_3 - x_4 &= 6 \\
x_2 + 4x_3 + x_5 &= 6 \\
\end{aligned}
\end{align*}
\]

3. Find the inverse of the matrix by using the rref of some matrix.

\[
A = \begin{bmatrix}
1 & -2 & -1 \\
-1 & 5 & 2 \\
5 & -4 & 5
\end{bmatrix}
\]

(Determinants, Cramer’s rule, Cofactor expansion)

1. Let \(A\) and \(B\) be 5 \times 5\ matrices, with \(\det A = -2\) and \(\det B = 3\).

Determine the following:

(a) \(\det(AB)\)

(b) \(\det(2B)\)

(c) \(\det(B^2)\)

(d) \(\det(B^{-1}AB)\)

2. Solve the linear system for the unknown \(z\). You don’t need to solve for the other unknowns.

\[
\begin{align*}
2x + 3y - 4z - u &= 1 \\
y - 2z + 2u &= 0 \\
z + 3u &= 0 \\
-z + u &= 1
\end{align*}
\]

3. (a) Evaluate the determinant of the following matrix by reducing it into the upper triangular form first.

\[
\begin{bmatrix}
1 & 3 & -2 & 3 \\
0 & 2 & -1 & 4 \\
1 & 2 & 0 & -2 \\
2 & 3 & -1 & 2
\end{bmatrix}
\]

(b) Evaluate the determinant of the same matrix as in (a) by using the cofactor expansion. Verify that both methods yield the same answer.

(Rank of a matrix, Dimension of a subspace)

1. Given the reduced row echelon form for matrix \(A\) answer each of the following:

(a) The null space of the matrix \(A\) is a subspace of \(\mathbb{R}^p\). What is \(p\)? Construct a basis for the null space of \(A\).

(b) Construct a basis for the column space of \(A\).

(c) The column space of the matrix \(A\) is a subspace of \(\mathbb{R}^q\). What is \(q\)? Find the dimension for the column space of \(A\). (Hint: use the Rank theorem)
1. Determine if the columns of the given matrix form a linearly dependent set. You need to justify your conclusion.

\[ A = \begin{bmatrix} 3 & 4 & 3 \\ -1 & -7 & 7 \\ 1 & -3 & -2 \\ 0 & 2 & -6 \end{bmatrix} \]

(b) Determine whether the vector \[
\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}
\] is in the column space of \( A \).

2. Given the reduced row echelon form of \( A \) work out each of the following parts. Don’t be sloppy in your work.

\[ A = \begin{bmatrix} 3 & -5 & -1 & 4 & 4 \\ -2 & 4 & 2 & 7 & 8 \\ 5 & -9 & -3 & -3 & -4 \\ -2 & 6 & 6 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & \frac{5}{2} \\ 0 & 1 & 2 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

(a) Find a basis for the null space of \( A \).
(b) What is the rank of \( A \)?
(c) What is the nullity of \( A \)?
(d) Determine the dimension for the column space of \( A \).
(e) State the Rank Theorem and verify it with this example.

3. Let \( S \) be the solution space of the following system

\[ \begin{align*}
2x + 3y - 4z - u + 3v - w &= 0 \\
2y - 2z + 2u + 2w &= 0 \\
x + 3u - v - w &= 0 \\
-z + u + 2v + w &= 0
\end{align*} \]

(a) \( S \) is a subspace of \( \mathbb{R}^p \). What is \( p \)?
(b) Find the dimension of \( S \).

(Spanning, Independence)

1. (a) Determine if the columns of the given matrix form a linearly dependent set. You need to justify your conclusion.

\[ A = \begin{bmatrix} 3 & 4 & 3 \\ -1 & -7 & 7 \\ 1 & 3 & -2 \\ 0 & 2 & -6 \end{bmatrix} \]
(c) Is the vector \[
\begin{bmatrix}
1 \\
2 \\
-1 \\
1
\end{bmatrix}
\] in the null space of \( A \)?

2. Is the vector \[
\begin{bmatrix}
1 \\
2 \\
-1
\end{bmatrix}
\] in the span of \[
\begin{bmatrix}
3 \\
4 \\
3
\end{bmatrix}, \begin{bmatrix}
-1 \\
-7 \\
7
\end{bmatrix}, \begin{bmatrix}
1 \\
3 \\
-2
\end{bmatrix}, \text{ and } \begin{bmatrix}
0 \\
2 \\
-6
\end{bmatrix}?
\]

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(Eigenvalues, Eigenvectors, Diagonalization)

1. Let \( A = \begin{bmatrix} 22 & -100 \\ 5 & -23 \end{bmatrix} \), \( v_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \), and \( v_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \). Suppose you are told that \( v_1 \) and \( v_2 \) are eigenvectors of \( A \).

   (a) Use this information to diagonalize \( A \).

   (b) Calculate \( A^{20} \).

2. Find the eigenvalues and the corresponding vectors of the given matrix.

   \[
   \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}
   \]

3. (a) Solve the difference equation \( x_{k+1} = Ax_k \), where

   \[
   A = \begin{bmatrix} 1.25 & -7.5 \\ -7.5 & 1.25 \end{bmatrix}
   \]

   and \( x_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \). We are given the eigenvalue and the corresponding eigenvector for the problem: \( \lambda_1 = 2, v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) and \( \lambda_2 = 0.5, v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

   (b) Classify the origin as an attractor, repeller, or saddle point of the dynamical system \( x_{k+1} = Ax_k \) in part (a).

   (c) Instead of solving a difference equation as in part (a) we now want to solving the differential system \( \dot{x} = Ax \) subject to the given initial condition \( x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \). Construct the unique solution to the initial value problem.

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(Projection, Orthogonality)

1. Find an orthogonal basis for the column space \( U \) of the matrix given below. (Hint: use the Gram-Schmidt process)
2. (a) Find the closest point \( \hat{y} \) to \( y \) in the subspace \( W \) spanned by \( v_1 \) and \( v_2 \). Note that since \( v_1 \) and \( v_2 \) are not orthogonal, you need to apply the Gram-Schmidt process to orthogonalize them first.

\[
\begin{bmatrix}
3 \\
1 \\
1
\end{bmatrix}, \begin{bmatrix}
-1 \\
-1 \\
1
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}
\]

(b) Find the shortest distance from \( y \) to the subspace \( W \) defined in (a).

(Certain Applications)

1. (a) Find a least-squares solution of the inconsistent system \( Ax = b \) for

\[
A = \begin{bmatrix}
4 & 0 \\
0 & 2 \\
1 & 1
\end{bmatrix}, \quad b = \begin{bmatrix}
2 \\
0 \\
11
\end{bmatrix}
\]

(b) Calculate the shortest distance for \( b \) to the column space of \( A \). The answer is a scalar.

2. Find the equation \( y = \beta_0 + \beta_1x \) of the least-squares line that best fits the data points \((2, 1), (5, 2), (7, 3), (8, 3), (9, 2), (10, 4)\).