1. (25 points) Determine if the columns of the given matrix form a linearly independent set. You need to justify your conclusion.

\[
\begin{bmatrix}
3 & 4 & -1 \\
5 & -1 & 6 \\
1 & 3 & -2 \\
0 & 2 & -2 \\
\end{bmatrix}
\]

(Solution: Let the column vectors of the matrix be denoted by \( \mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} \), \( \mathbf{v}_2 = \begin{bmatrix} 4 \\ -1 \\ 3 \\ 2 \end{bmatrix} \), and \( \mathbf{v}_3 = \begin{bmatrix} -1 \\ 6 \\ -2 \\ -2 \end{bmatrix} \). We need to solve the vector equation \( c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0} \) for \( c_1, c_2, \) and \( c_3 \).

The augmented matrix is

\[
\begin{bmatrix}
3 & 4 & -1 & 0 \\
5 & -1 & 6 & 0 \\
1 & 3 & -2 & 0 \\
0 & 2 & -2 & 0 \\
\end{bmatrix}
\]

The reduced row echelon form is

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Thus we solve the corresponding system \( \begin{cases} c_1 + c_3 = 0 \\ c_2 - c_3 = 0 \end{cases} \) using the back substitution. This leads to \( c_3 = t, c_2 = t, \) and \( c_1 = -t \). We see that the system has nontrivial solutions. For instance, setting \( t = 1 \) we get \( c_3 = 1, c_2 = 1, \) and \( c_1 = -1 \) which is a nontrivial solution to the vector equation \( c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0} \). We conclude that \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) are linearly dependent.

2. (25 points) Use Kirchhoff’s Law to write a matrix equation that determines the loop currents \( I_1, I_2, \) and \( I_3 \) for the following circuit. You don’t need to solve the system.

(Solution: Since I have technical difficulties to import a graph into the document, we need to refer to the circuit printed on the quiz. Apply the Kirchhoff’s rule to the third loop and note that the potential drop across the battery in the orientation of \( I_3 \) is considered as positive. Thus we get

\[4(\mathbb{I}_3 - \mathbb{I}_2) + (\mathbb{I}_3 - \mathbb{I}_1) + 12 = 0 \Rightarrow \mathbb{I}_1 + 4\mathbb{I}_2 - 5\mathbb{I}_3 = 12.\]

In a similar way from the first loop we get

\[(\mathbb{I}_1 - \mathbb{I}_3) + 2(\mathbb{I}_1 - \mathbb{I}_2) + 2\mathbb{I}_1 = 0 \Rightarrow 5\mathbb{I}_1 - 2\mathbb{I}_2 - \mathbb{I}_3 = 0.\]

Finally from the second loop we obtain
\[ 2I_1 - 9I_2 + 4I_3 = 0. \] Putting them together we have the system
\[
\begin{align*}
5I_1 - 2I_2 - I_3 &= 0 \\
2I_1 - 9I_2 + 4I_3 &= 0 \\
I_1 + 4I_2 - 5I_3 &= 12
\end{align*}
\]

In term of matrix notation we have
\[
\begin{bmatrix}
5 & -2 & -1 \\
2 & -9 & 4 \\
1 & 4 & -5
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
12
\end{bmatrix}
\]

This is a matrix equation for the problem. There are some other equivalent matrix equations. But those are just the trivial variants of the above equation.)