(1) (2) (3) (4) (5) (6) (7) (8) Raw: ____ Final: ____

Note: You can have a Raw Score of up to 140 points on this exam. Your Final Score will be your Raw Score divided by 1.3, so your Final Score can be up to 108 points. Any Final Score over 100 will be considered extra credit. Calculators are not allowed on this exam. YOU MUST SHOW ALL WORK TO RECEIVE CREDIT.

10 pts (1) Given \( f(x) = \sqrt{x + 3} \), find a formula for \( f^{-1}(x) \) and state the domain and range of \( f^{-1} \). Make sure you clearly indicate which part of your answer is the domain and which is the range. To receive full credit, you must use interval notation.

First find domain and range of \( f(x) \).

Domain: \( x + 3 \geq 0 \Rightarrow x \geq -3 \) \( \left[ -3, \infty \right) \)

Range: \( y \geq 0 \) \( \left[ 0, \infty \right) \)

Find the inverse:
\[ y = \sqrt{x + 3} \Rightarrow y^2 = x + 3 \]
\[ y^2 - 3 = x \Rightarrow f^{-1}(y) = x^2 - 3 \]

Domain of \( f^{-1} = \) range of \( f = \left[ 0, \infty \right) \)

Range of \( f^{-1} = \) domain of \( f = \left[ -3, \infty \right) \)

10 pts (2) Prove that if \( f(x) = \sec(x) \), then \( f'(x) = \sec(x) \tan(x) \). You must show all the steps to receive credit.

\[ f(x) = \sec x = \frac{1}{\cos x} \]

\[ f'(x) = \frac{(\cos x)(1)' - (1)(\cos x)'}{(\cos x)^2} = \frac{0 - \sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} \]

\[ = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x \]
15 pts (3) On the axes below, sketch a possible graph of the function $f(x)$ such that all of the following properties hold:

**Domain of $f$ is $(\infty, \infty)$**

$f(-1) = f(1) = f(3) = 3$

\[
\lim_{x \to -\infty} f(x) = 5
\]

\[
\lim_{x \to -1} f(x) = \infty
\]

\[
\lim_{x \to \infty} f(x) = -\infty
\]

$f$ is continuous at $x = 1$, but $f'(1)$ does not exist.

$f$ is not continuous at $x = 3$, but $f$ is continuous from the right at $x = 3$.

This is one possibility
10 pts (4) Find \( f'(x) \) given \( f(x) = x^2 + 3x + 2 \). You MUST use the definition of a derivative.

\[
    f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 3(x+h) + 2 - (x^2 + 3x + 2)}{h}
\]

\[
= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h + 2 - x^2 - 3x - 2}{h}
\]

\[
= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h} = \lim_{h \to 0} \frac{h(2x + h + 3)}{h} = \lim_{x \to 0} 2x + h + 3
\]

\[
= 2x + 0 + 3 = 2x + 3
\]

10 pts (5) Prove that the stated formula is the local linear approximation near \( x_0 = 1 \).

\[
\sqrt{x} \approx 1 + \frac{1}{2} (x - 1)
\]

\[
f(x) \approx f(x_0) + f'(x_0)(x-x_0)
\]

\[
f'(x) = \frac{1}{2\sqrt{x}}
\]

\[
\boxed{\begin{align*}
\text{(1)} & \quad f(x_0) = f(1) = \sqrt{1} = 1 \\
\text{(2)} & \quad f'(x_0) = f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}
\end{align*}}
\]

Plugging in, we get \( \sqrt{x} \approx 1 + \frac{1}{2} (x-1) \)
55 pts (6) Find $y'(x)$ given:

5 pts (a) $y = \sqrt{\ln(x)} = (\ln x)^{\frac{1}{2}}$

$$y' = \frac{1}{2}(\ln x)^{-\frac{1}{2}} \cdot (\ln x)' = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

5 pts (b) $y = \ln(\log_7 x)$

$$y' = \frac{1}{\log_7 x} \cdot (\log_7 x)' = \frac{1}{\log_7 x} \cdot \frac{1}{\ln 7} \cdot \frac{1}{x} = \frac{1}{x(\ln 7) \log_7 x}$$

5 pts (c) $y = e^{x \cdot \tan(x)}$

$$y' = e^{x \tan x} \cdot (x \tan x)' = e^{x \tan x} \cdot [x \tan(x)' + \tan x \cdot (x)'] = e^{x \tan x} \cdot [x \sec^2 x + \tan x]$$

5 pts (d) $y = \frac{1}{\arccos(x)} = (\arccos x)^{-1}$

$$y' = -\frac{1}{(\arccos x)^2} \cdot (\arccos(x))' = -\frac{1}{(\arccos x)^2} \cdot \frac{-1}{\sqrt{1-x^2}} = \frac{1}{(\arccos x)^2 \sqrt{1-x^2}}$$
55 pts (6) (continued) Find $y'(x)$ given:

5 pts (e) $y = \tan^{-1}(\sin^{-1}x)$

$$y' = \frac{1}{1 + (\sin^{-1}x)^2} \cdot (\sin^{-1}x)' = \frac{1}{1 + (\sin^{-1}x)^2} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$y' = \frac{1}{(1 + (\sin^{-1}x)^2)(\sqrt{1-x^2})}$$

10 pts (f) $y = x^5 \sec\left(\frac{1}{x}\right)$

$$y' = x^5 (\sec\left(\frac{1}{x}\right))' + \sec\left(\frac{1}{x}\right) (x^5)'$$

$$y' = x^5 (\sec\left(\frac{1}{x}\right))(\tan\left(\frac{1}{x}\right)) \cdot \left(\frac{1}{x^2}\right)' + \sec\left(\frac{1}{x}\right) \cdot 5x^4$$

$$y' = x^5 (\sec\left(\frac{1}{x}\right))(\tan\left(\frac{1}{x}\right)) \cdot \left(-\frac{1}{x^2}\right) + 5x^4 \sec\left(\frac{1}{x}\right)$$

$$y' = 5x^4 \sec\left(\frac{1}{x}\right) - x^3 (\sec\left(\frac{1}{x}\right))(\tan\left(\frac{1}{x}\right))$$

$$y' = x^3 \sec\left(\frac{1}{x}\right) \left[ 5x - \tan\left(\frac{1}{x}\right) \right]$$

10 pts (g) $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = 1$

Hint: Use implicit differentiation

$$x^{-3/2} + y^{-3/2} = 1$$

$$\left(x^{-3/2} + y^{-3/2}\right)' = (1)'$$

$$-\frac{1}{2}x^{-5/2} - \frac{1}{2}y^{-5/2} y' = 0$$

$$-\frac{1}{2}x^{-3/2} = \frac{1}{2}y^{-3/2} y'$$

$$y' = \frac{-y^{-3/2}}{x^{3/2}}$$
55 pts (6) (continued) Find \( y'(x) \) given:

\[ y = \frac{\sin(x) \cdot \cos(x) \cdot \tan^3(x)}{\sqrt{x}} \]

\[ \ln y = \ln \left[ \frac{\sin x \cdot \cos x \cdot \tan^3 x}{\sqrt{x}} \right] = \ln \sin x + \ln \cos x + 3 \ln \tan x - \frac{1}{2} \ln x \]

Take derivative of both sides with respect to \( x \)

\[ \frac{1}{y} y' = \frac{1}{\sin x} \cdot \cos x + \frac{1}{\cos x} (-\sin x) + \frac{3}{\tan x} \sec^2 x - \frac{1}{2x} \]

\[ y' = [\cot x - \tan x + 3 \sec^2 x - \frac{1}{2x}] \left( \frac{\sin x \cdot \cos x \cdot \tan^3 x}{\sqrt{x}} \right) \]

10 pts (7) Find the equation of the tangent line to \( y = 3 \cdot \cot^4(x) \) at \( x = \frac{\pi}{4} \).

First find \( m_{\text{tan}} = y' \)

\[ y' = 3 \cdot 4 \cdot \cot^3 x \cdot (-\csc^2 x) = 12 \cot^3 x (-\csc^2 x) \]

\[ y' \left( \frac{\pi}{4} \right) = -12 \left( \cot \left( \frac{\pi}{4} \right) \right)^3 \left( \csc \left( \frac{\pi}{4} \right) \right)^2 \]

\[ \cot \left( \frac{\pi}{4} \right) = \frac{\cos \left( \frac{\pi}{4} \right)}{\sin \left( \frac{\pi}{4} \right)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 \]

\[ \csc \left( \frac{\pi}{4} \right) = \frac{1}{\sin \left( \frac{\pi}{4} \right)} = \frac{\sqrt{2}}{\frac{\sqrt{2}}{2}} = 2 = \sqrt{2} \]

\[ y = y_0 = m \cdot (x-x_0) \quad y_0 = y \left( \frac{\pi}{4} \right) = 3 \left( \cot \left( \frac{\pi}{4} \right) \right)^4 = 3 \cdot 1^4 = 3 \]

\[ y - 3 = -24 \left( x - \frac{\pi}{4} \right) \]

\[ y = -24x + 6\pi + 3 \]
20 pts (8) Find the following limits. If you use L'Hopital's Rule, you must indicate the indeterminate form to receive full credit.

5 pts (a) \[ \lim_{x \to \infty} \frac{(\sqrt{x}) \cdot (x^2 + 1)}{\infty \cdot \infty} = \infty \] Not an indeterminate form.

5 pts (b) \[ \lim_{x \to 0} \frac{e^x - 1}{\sin(x)} \to 0 \] Indeterminate form \((\frac{0}{0})\)

\[ = \lim_{x \to 0} \frac{e^x}{\cos x} = \frac{e^0}{\cos 0} = \frac{1}{1} = 1 \]

10 pts (c) \[ \lim_{x \to 0} \left( e^x + x \right)^{\frac{1}{x}} \] Indeterminate form \((1)^\infty\)

let \( y = \left( e^x + x \right)^{\frac{1}{x}} \). We want \( \lim_{x \to \infty} y \). First find \( \lim_{x \to \infty} \ln y \).

\[ \ln y = \ln \left( e^x + x \right)^{\frac{1}{x}} = \frac{1}{x} \ln \left( e^x + x \right) = \frac{\ln (e^x + x)}{x} \]

\[ \lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\ln (e^x + x)}{x} \to 0 \] Indeterminate form \((\frac{0}{0})\)

\[ = \lim_{x \to 0^+} \frac{1}{e^x + x} \cdot e^x + x = \lim_{x \to 0^+} \frac{e^x + 1}{e^x + 0} = \frac{e^0 + 1}{e^0 + 0} = \frac{2}{1} = 2 \]

So \( \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^2 \)