SLOPE OF TANGENT LINE TO THE
GRAPH OF $f$ AT $(x, f(x))$

= INSTANTANEOUS RATE OF CHANGE
OF $f$ AT $x$

$$= \lim_{{h \to 0}} \frac{f(x+h)-f(x)}{h} \quad \text{(PROVIDED THE LIMIT EXISTS)}$$

ANGE:

$\frac{\text{NEW STUFF}}{\text{THE Derivative of $f$ at $x$}}$

$= f'(x)$

$= y'$

$= \frac{dy}{dx}$

$= \frac{d}{dx} (f(x))$

EXAMPLE: LET $f(x) = \sqrt{x}$ FOR $x > 0$. THEN

$$\lim_{{h \to 0}} \frac{f(x+h)-f(x)}{h} = \lim_{{h \to 0}} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{{h \to 0}} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{{h \to 0}} \frac{(x+h) - x}{h (\sqrt{x+h} + \sqrt{x})} = \lim_{{h \to 0}} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad (\text{PROVIDED $x \neq 0$})$$

THUS,

$$f'(x) = \frac{1}{2\sqrt{x}} \quad \text{for} \quad x > 0$$
THE PROCESS OF FINDING DERIVATIVES OF FUNCTIONS IS CALLED

**DIFFERENTIATION**

**IF A FUNCTION HAS A DERIVATIVE AT A POINT IT IS SAID TO BE**

**DIFFERENTIABLE**

**AT THAT POINT**

E.C., \( f(x) = \sqrt{x} \) **IS DIFFERENTIABLE AT EVERY**

**POINT** **IN ITS DOMAIN EXCEPT** \( x = 0 \).

**GEOMETRICAL REASON:**

![Diagram of \( y = f(x) = \sqrt{x} \)]

**VER vertical TANGENT**

**T (UND EFINED**

**SLOPE)**

**ANOTHER EXAMPLE:** \( f(x) = |x| \) **IS NOT DIFFERENTIABLE AT** \( x = 0 \)

\[
\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}
\]

**WHICH DOES NOT EXIST** **BECAUSE**

\[
\frac{|h|}{h} = \begin{cases} 
1, & h > 0 \\
-1, & h < 0
\end{cases}
\]
ANOTHER WAY FOR DIFFERENTIABILITY TO FAIL:

**Theorem:** If \( f(x) \) is differentiable at \( x_0 \), then \( f(x) \) is continuous at \( x_0 \).

(So, if \( f(x) \) is not continuous at \( x_0 \), then \( f(x) \) is not differentiable at \( x_0 \)).

**Reason:** If \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \), then

\[
\lim_{x \to x_0} \left( f(x) - f(x_0) \right) = \lim_{x \to x_0} \left( \frac{f(x) - f(x_0)}{x - x_0} \right) (x - x_0)
\]

\[
= \left( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \right) \left( \lim_{x \to x_0} (x - x_0) \right)
\]

\[
= f'(x_0) \cdot 0
\]

\[
= 0
\]

So

\[
\lim_{x \to x_0} f(x) = f(x_0).
\]

Here, then, are the ways in which \( f(x) \) can fail to be differentiable at \( x_0 \):

\[\text{Graphs of differentiability failures.}\]
MORE EXAMPLES:

1. Compute \( \frac{dy}{dx} \) if \( y = f(x) = x^2 + 2x - 1 \)

\[
\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) - 1 - (x^2 + 2x - 1)}{h}
\]

\[
= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h - 1 - x^2 - 2x + 1}{h}
\]

\[
= \lim_{h \to 0} \frac{h(2x + h + 2)}{h} = \lim_{h \to 0} (2x + h + 2) = 2x + 2
\]

2. Find the equation of the tangent line to the graph of \( y = f(x) = x^2 + 2x - 1 \)

at the point \((1,2)\).

Point-slope form: \( y - y_0 = m(x - x_0) \)

\[ \begin{align*}
2 & \quad \downarrow \quad \downarrow \\
\text{Slope of tangent line at } (1,2) & \quad \\
\end{align*} \]

\[ f'(1) = 2(1) + 2 \quad \text{(from example 1)} \]

\[ = 4 \]

\[ y - 2 = 4(x - 1) \]

\[ y = 4x - 2 \]
3. **Find the point on the graph of** \( y = f(x) = x^2 + 2x - 1 \)**. **Where the tangent line is horizontal.**

**Tangent line horizontal** \( \iff \) **Slope of tangent line zero** \( \iff \) **Derivative zero** \( \iff f'(x) = 0 \)

\[ 2x + 2 = 0 \quad \text{(from Example 1)} \]
\[ x = -1 \]
\[ y = f(-1) = (-1)^2 + 2(-1) - 1 \]
\[ = -2 \]

**Answer:** \((-1, -2)\)

4. **Compute** \( g'(t) \) **if** \( g(t) = \frac{1}{t} \)

\[ g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h} \]

\[ = \lim_{h \to 0} \frac{\frac{1}{t+h} - \frac{1}{t}}{h} \]

\[ = \lim_{h \to 0} \frac{\frac{t - (t+h)}{t(t+h)}}{h} = \lim_{h \to 0} \frac{-1}{t(t+h)} \]

\[ = -\frac{1}{t^2} \]