All of these problems are to be worked out in detail. You must show all your work on this paper to receive full credit.

1. (12 pts) Find the arc length of the curve \( y = 2x^{\frac{3}{2}} \) from \( x = 0 \) to \( x = 1 \). Give your answer in exact form and show all steps in the calculation.

Solution. Since \( \frac{dy}{dx} = 2(\frac{3}{2}x^{\frac{1}{2}}) = 3\sqrt{x} \), the arc length is given by the integral

\[
L = \int_{0}^{1} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_{0}^{1} \sqrt{1 + (3\sqrt{x})^2} \, dx = \int_{0}^{1} \sqrt{1 + 9x} \, dx
\]

Using the change of variable \( u = 1 + 9x \) (\( du = 9dx \)) it follows that

\[
L = \int_{0}^{1} \sqrt{1 + 9x} \, dx = \frac{1}{9} \int_{1}^{10} \sqrt{u} \, du = \frac{1}{9} \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_{1}^{10} = \frac{2}{27} (10\sqrt{10} - 1) \approx 2.268 \ldots
\]

2. (12 pts) A spring stretches 6 inches beyond its natural length under a force of 50 pounds. How much work is done in stretching it 1 additional foot (i.e. from 6 inches to 18 inches)?

Solution. From Hooke’s Law, the force required to stretch the spring \( x \) units is proportional to \( x \), i.e. \( f(x) = kx \). In this problem we have \( f(\frac{1}{2}) = 50 \), and so the spring constant is \( k = 100 \). Thus the work done in stretching the spring 1 additional foot is

\[
W = \int_{\frac{1}{2}}^{\frac{3}{2}} 100x \, dx = 50x^2 \bigg|_{\frac{1}{2}}^{\frac{3}{2}} = 50\left(\frac{9}{4} - \frac{1}{4}\right) = 50(2) = 100 \text{ ft-lbs}
\]
3. (10 pts) The rectangular tank shown here, with its top at ground level, is used to catch runoff water. If the tank is full, how much work is required to empty it by pumping the water back to ground level? (Use the fact that water weighs 62.5 lb/ft$^3$.)

**Solution.** The volume of a thin horizontal slice of water of thickness $\Delta y = dy$ can be represented by

$$dV = (\text{area}) \Delta y = (12)(8) \Delta y = 96dy$$

and the weight (in pounds) of such a slice is

$$(62.5)dV = 6000dy$$

The slice at depth $y$ must be raised a distance of $y$ feet to the top of the tank. Thus the amount of work required to empty the tank is

$$W = \int_0^{15} (y)(6000dy) = \int_0^{15} 6000y dy = 3000y^2 \bigg|_0^{15} = 675000 \text{ ft-lbs}$$

4. For the tank pictured in Problem 3: **Set up an integral** that represents the hydrostatic force on the front (12 ft by 15 ft) side of the tank when the tank is full. (Use the fact that the pressure at depth $y$ is 62.5y lb/ft$^2$.) DO NOT EVALUATE THE INTEGRAL.

**Solution.** The area of a thin horizontal strip (across the front of the tank) is $dA = 12dy$, and the force on the strip at depth $y$ is represented by

$$(62.5y)dA = 750y dy$$

Thus the total hydrostatic force on the front of the tank is given by the integral

$$F = \int_0^{15} 750y dy$$

Here, for no extra charge (or credit), is the evaluation of the integral:

$$F = \int_0^{15} 750y dy = 375y^2 \bigg|_0^{15} = 375(15)^2 = 84375 \text{ lbs}$$