Some formulas:

\[ M_y = \rho \int_a^b xf(x) \, dx \]

\[ M_x = \rho \int_a^b \frac{1}{2} [f(x)]^2 \, dx \]

\[ \bar{x} = \frac{1}{A} \int_a^b xf(x) \, dx \]

\[ \bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} [f(x)]^2 \, dx \]

Part I. The problems on this part are of multiple-choice type. Circle the correct answer. (5 pts each)

1. Find the coordinates of the center of mass of the discrete system of mass particles consisting of a mass \( m_1 = 1 \) at (1,2), a mass \( m_2 = 2 \) at (2,0), and a mass \( m_3 = 3 \) at (–1,–1).

A) \( \left( \frac{1}{3}, \frac{1}{6} \right) \)

B) \( \left( \frac{1}{3}, \frac{1}{6} \right) \)

C) \( \left( -\frac{1}{3}, \frac{1}{6} \right) \)

D) \( \left( \frac{1}{6}, \frac{1}{3} \right) \)

E) \( \left( \frac{1}{6}, \frac{1}{3} \right) \)

F) \( \left( -\frac{1}{6}, \frac{1}{3} \right) \)

Solution.

\[
\bar{x} = \frac{M_y}{m} = \frac{(1)(1) + (2)(2) + (3)(-1)}{1 + 2 + 3} = \frac{1}{3} \quad \bar{y} = \frac{M_x}{m} = \frac{(1)(2) + (2)(0) + (3)(-1)}{1 + 2 + 3} = -\frac{1}{6}
\]

The correct answer is B.

2. Find the coordinates of the center of mass of the flat plate (lamina) in the figure on the right. Assume that the plate has uniform density \( \rho = 1 \).

A) \( \left( \frac{1}{2}, \frac{5}{8} \right) \)

B) \( \left( \frac{1}{2}, \frac{3}{4} \right) \)

C) \( \left( \frac{1}{2}, \frac{7}{8} \right) \)

D) \( \left( \frac{1}{2}, \frac{11}{16} \right) \)

E) \( \left( \frac{1}{2}, \frac{13}{16} \right) \)

F) \( \left( \frac{3}{4}, \frac{1}{2} \right) \)

Solution. The center of mass of the lamina is the same as the center of mass of the two-point mass system consisting of a mass \( m_1 = 1 \) at \( \left( \frac{1}{2}, \frac{3}{2} \right) \), and a mass \( m_2 = 3 \) at \( \left( \frac{1}{2}, \frac{1}{2} \right) \). Thus \( \bar{x} = \frac{1}{2} \) (symmetry) and

\[
\bar{y} = \frac{(1)(\frac{3}{2}) + (3)(\frac{1}{2})}{1 + 3} = \frac{3}{4}.
\]

The correct answer is B.
3. Which of these functions is a solution of the differential equation \( \frac{dy}{dx} + 2xy = 0 \)?

A) \( y = e^x \)  
B) \( y = e^{-x} \)  
C) \( y = e^{x^2} \)

D) \( y = e^{-x^2} \)  
E) \( y = e^{2x} \)  
F) \( y = e^{-2x} \)

**Solution.** If \( y = e^{-x^2} \), then \( \frac{dy}{dx} = e^{-x^2}(-2x) = -2xe^{-x^2} \) and so

\[
\frac{dy}{dx} + 2xy = (-2xe^{-x^2}) + 2x(e^{-x^2}) = -2xe^{-x^2} + 2xe^{-x^2} = 0
\]

The correct answer is D.

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**Part II. The problems on this part are to be worked out in detail.** You must show all your work on this paper to receive full credit.

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4. (12 pts) Find the centroid of the region under the graph of \( y = x^2 + 1 \) for \(-1 \leq x \leq 1\) (see figure). [Consider the region as representing a lamina of uniform density \( \rho = 1 \). Feel free to use symmetry.]

**Solution.** From the symmetry principle, \( \bar{x} = 0 \).

\[
A = 2\int_{0}^{1} (x^2 + 1)dx = 2\left(\frac{1}{3}x^3 + x\right)_{0}^{1} = 2\left(\frac{1}{3} + 1\right) = \frac{8}{3}
\]

\[
\bar{y} = \frac{1}{A} \left[ 2\int_{0}^{1} \frac{1}{2} (x^2 + 1)^2dx \right] = \frac{3}{8} \int_{0}^{1} (x^2 + 1)^2dx = \frac{3}{8} \int_{0}^{1} (x^4 + 2x^2 + 1)dx
\]

\[
= \frac{3}{8} \left[ \frac{1}{5}x^5 + \frac{2}{3}x^3 + x \right]_{0}^{1} = \frac{3}{8} \left[ \frac{1}{5} + \frac{2}{3} + 1 \right] = \frac{3}{8} \left[ \frac{28}{15} \right] = \frac{7}{10}
\]

The centroid of the region is at the point \( (0, \frac{7}{10}) \).
5. (13 pts) Find the solution of the differential equation \( \frac{dy}{dx} = \frac{\cos(x)}{y} \) that satisfies the initial condition \( y(0) = -1 \). Express the solution explicitly as a function of \( x \), i.e. in the form \( y = y(x) \).

**Solution.** Upon separating variables the equation can be written in the form \( y \, dy = \cos(x) \, dx \), and we can then solve it by integrating both sides independently.

\[
\int y \, dy = \int \cos(x) \, dx
\]

\[
\frac{1}{2} y^2 = \sin(x) + C
\]

\[
y = \pm \sqrt{2 \sin(x) + 2C}
\]

From the condition \( y(0) = -1 \) we conclude that the – sign represents the appropriate branch of the general solution, and that \( -1 = -\sqrt{0 + 2C} \). Thus \( C = \frac{1}{2} \) and the solution of the initial value problem is

\[
y = -\sqrt{2 \sin(x) + 1}
\]