THE FUNDAMENTAL THEOREM OF CALCULUS:

THERE ARE TWO PARTS TO THIS. AS MOTIVATION FOR THE FIRST, RECALL THAT

\[ y = f(x) \]

\[ A(x) \]

\[ a \quad x \quad b \]

\[ A'(x) = f(x) \]

\[ \frac{d}{dx} A(x) = f(x) \]

\[ \frac{d}{dx} \int_{a}^{x} f(t)dt = f(x) \]

THE FUNDAMENTAL THEOREM OF CALCULUS SAYS THAT THIS IS ALWAYS TRUE:

IF \( f(x) \) IS CONTINUOUS ON THE INTERVAL \( I \) AND \( a \) IS IN \( I \), THEN

\[ \frac{d}{dx} \int_{a}^{x} f(t)dt = f(x) \]

I.E., \( F(x) = \int_{a}^{x} f(t)dt \) IS AN ANTIDERIVATIVE FOR \( f(x) \).
EXAMPLES:

1. \( \frac{d}{dx} \int_{1}^{x} \cos t \, dt = \cos x \)

2. \( \frac{d}{dx} \int_{5}^{x} \cos t \, dt = \cos x \)

3. DEFINE A FUNCTION \( y = f(x) \) BY

\[
f(x) = \int_{\sqrt{3}}^{x} \arctan t \, dt.
\]

FIND \( f(\sqrt{3}), f'(\sqrt{3}) \) AND \( f''(\sqrt{3}) \).

\[
f(\sqrt{3}) = \int_{\sqrt{3}}^{\sqrt{3}} \arctan t \, dt = 0
\]

\[
f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \int_{\sqrt{3}}^{x} \arctan t \, dt
\]

\[
= \arctan x
\]

\[
f'(\sqrt{3}) = \arctan \sqrt{3} = \frac{\pi}{3}
\]

\[
f''(x) = (\arctan x)' = \frac{1}{1+x^2}
\]

\[
f''(\sqrt{3}) = \frac{1}{1+(\sqrt{3})^2} = \frac{1}{4}
\]

4. \( \frac{d}{dx} \int_{x}^{1} \frac{\sin t}{t} \, dt = \frac{d}{dx} \left( - \int_{1}^{x} \frac{\sin t}{t} \, dt \right) \)

\[
= - \frac{d}{dx} \int_{1}^{x} \frac{\sin t}{t} \, dt = - \frac{\sin x}{x}\]
THE OTHER HALF OF THE FUNDAMENTAL THEOREM OF CALCULUS IS
VERY CLOSELY RELATED TO THIS.

SUPPOSE $f(x)$ IS CONTINUOUS ON $[a, b]$. THEN $\int_a^x f(t)\,dt$
IS AN ANTIDERIVATIVE FOR $f(x)$ ON $[a, b]$.

ANY OTHER ANTIDERIVATIVE $F(x)$ FOR $f(x)$ ON $[a, b]$ DIFFERS
FROM THIS ONE BY SOME CONSTANT:

$$F(x) = \int_a^x f(t)\,dt + C$$

NOW NOTICE THAT

$$F(b) - F(a) = \left[ \int_a^b f(t)\,dt + C \right] - \left[ \int_a^a f(t)\,dt + C \right]
= \int_a^b f(t)\,dt + C - 0 - C
= \int_a^b f(x)\,dx$$

IF $f(x)$ IS CONTINUOUS ON $[a, b]$
AND $F(x)$ IS ANY ANTIDERIVATIVE
FOR $f(x)$ ON $[a, b]$, THEN

$$\int_a^b f(x)\,dx = F(b) - F(a)
= F(x)\Big|_a^b$$
EXAMPLES:

1. \[ \int_{-1}^{1} \frac{1}{2} \cdot 4^y \, dx = \frac{1}{2} \left( 2^{1} - 2^{-1} \right) = \frac{1}{2} (14) = \frac{7}{2} \]

2. \[ \int_{0}^{2\pi} \cos x \, dx = \sin x \bigg|_{0}^{2\pi} = \sin 2\pi - \sin 0 = 0 - 0 = 0 \]

3. \[ \int_{-\frac{1}{2}}^{\frac{1}{2}} \arcsin x \, dx \]
   \[ = \left[ \arcsin x \right]_{-\frac{1}{2}}^{\frac{1}{2}} = \arcsin \frac{1}{2} - \arcsin \left(-\frac{1}{2}\right) \]
   \[ = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3} \]

4. \[ \int_{0}^{\frac{\pi}{4}} \tan x \, dx = \ln |\sec x| \bigg|_{0}^{\frac{\pi}{4}} = \ln |\sec \frac{\pi}{4}| - \ln |\sec 0| \]
   \[ = \ln |\sqrt{2}| - \ln 1 = \ln \sqrt{2} = \frac{1}{2} \ln 2 \]

5. \[ \int_{-e}^{e} \frac{1}{x} \, dx = \ln |x| \bigg|_{-e}^{e} = \ln |e| - \ln |-e| = \ln 1 - \ln e \]
   \[ = 0 - 1 = -1 \]

6. FIND THE AREA UNDER THE GRAPH OF \( y = \frac{1}{2} x^2 \) FROM \( x = -1 \) TO \( x = 1 \).

\[ A = \int_{-1}^{1} \frac{1}{2} x^2 \, dx = \arctan x \bigg|_{-1}^{1} = \arctan (1) - \arctan (-1) \]
   \[ = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2} \]
Notice that the following is obviously stupid:

\[
\int_{-1}^{1} \frac{1}{x^2} \, dx = -\frac{1}{x} \bigg|_{-1}^{1} = -\frac{1}{1} - (-\frac{1}{1}) = -1 - 1 = -2
\]

Why is it obviously stupid and what went wrong?

Recall the Mean Value Theorem (for derivatives) from last term:

\[
y = f(x), \text{ continuous on } [a, b] \text{ and differentiable on } (a, b)
\]

For some \( c \) in \( (a, b) \),

\[
f'(c) = \frac{f(b) - f(a)}{b-a}
\]

There is a similar Mean Value Theorem (for integrals):

\[
y = f(x), \text{ continuous on } [a, b]
\]

For some \( x^* \) in \( [a, b] \),

\[
\int_{a}^{b} f(x) \, dx = f(x^*) (b-a)
\]