\[ A = \pi r^2 \]
Suppose you were asked to find the area under a curve of arbitrary shape over the interval: \([a, b]\). How could you at least get a decent approximation?

A simple and effective way to estimate the area under the curve is to use the sum of the areas of small rectangles, all of the same width, with heights that vary as the curve.
Area Approximation Techniques

- Left Endpoint
- Right Endpoint
- Midpoint
Left Endpoint Approximation

\[ L_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x \]

\[ \Delta x = \frac{b - a}{n} \quad x_i = a + (i - 1) \cdot \Delta x \]
**Left Endpoint Approximation Example**

\[ f(x) := x^2 \quad (1, 5) \]

\[ \Delta x = \frac{(5 - 1)}{4} = 1 \]

\[ L_4 = 1 \cdot \left[ \frac{3}{2} \right] + \left[ \frac{3}{2} \right] + \left[ \frac{3}{2} \right] + \left[ \frac{3}{2} \right] \]
Right Endpoint Approximation

\[ R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x \]

\[ \Delta x = \frac{b - a}{n} \quad \quad x_i = a + i \cdot \Delta x \]
Right Endpoint Approximation

Example

\[ f(x) = x^2 \quad (1, 5) \]

\[ \Delta x = \frac{(5 - 1)}{4} = 1 \]

\[ R_4 = 1 \cdot \left[ \frac{3}{2} \cdot (2)^2 + \frac{3}{2} \cdot (3)^2 + \frac{3}{2} \cdot (4)^2 + \frac{3}{2} \cdot (5)^2 \right] \]
Midpoint Approximation

\[ M_n = \sum_{i = 1}^{n} f(x_i) \cdot \Delta x \]

\[ \Delta x = \frac{b - a}{n} \]
\[ x_i = a + \left( i - \frac{1}{2} \right) \cdot \Delta x \]
Midpoint Approximation

Example

\[ f(x) = x^2 \quad (1, 5) \]

\[ \Delta x = \frac{(5 - 1)}{4} = 1 \]

\[ M_4 = 1 \cdot \left[ (1.5)^2 + (2.5)^2 + (3.5)^2 + (4.5)^2 \right] \]
AREA APPROXIMATION
EXAMPLE / PART 1

Estimate the area under the parabola \( y = x^2 \) over the interval: \([0, 1]\).

The interval is 1 unit wide. Suppose we use 4 rectangles. Then each of the 4 rectangles must have a width of \(1/4\).

Choose the height of each rectangle to be the value of "y" at the left or right endpoints of the subintervals or anywhere within.

Each of the four rectangles approximates the area under the curve on one of 4 subintervals: \([0, .25], [.25, .5], [.5, .75], \& [.75, 1]\).
Now let's calculate the areas of the approximating rectangles using left and right endpoint approximations and compare those estimates to the actual area, which is "1/3".

\[
L := \frac{1}{4} \left[ 0^2 + \left( \frac{1}{4} \right)^2 + \left( \frac{1}{2} \right)^2 + \left( \frac{3}{4} \right)^2 \right] \quad \text{L} = 0.219
\]

\[
R := \frac{1}{4} \left[ \left( \frac{1}{4} \right)^2 + \left( \frac{1}{2} \right)^2 + \left( \frac{3}{4} \right)^2 + 1^2 \right] \quad \text{R} = 0.469
\]
Our estimates improve as we increase the number of strips. Suppose we divide the same interval into 6 subintervals.

\[
R := \frac{1}{6} \left[ \left( \frac{1}{6} \right)^2 + \left( \frac{2}{6} \right)^2 + \left( \frac{3}{6} \right)^2 + \left( \frac{4}{6} \right)^2 + \left( \frac{5}{6} \right)^2 + \left( \frac{6}{6} \right)^2 \right]
\]

\[
R = 0.421
\]

\[
L := \frac{1}{6} \left[ \left( \frac{0}{6} \right)^2 + \left( \frac{1}{6} \right)^2 + \left( \frac{2}{6} \right)^2 + \left( \frac{3}{6} \right)^2 + \left( \frac{4}{6} \right)^2 + \left( \frac{5}{6} \right)^2 \right]
\]

\[
L = 0.255
\]

Our left endpoint estimate is increased while our right endpoint estimate is decreased as the number of subintervals was increased.
Our estimates get more accurate as we increase the number of strips. They converge to a number between the left and right endpoints as the last graph suggests.

Suppose we use "n" strips. Then rectangles are each "1/n" units wide. The subintervals are: [0,1/n], [1/n,2/n]...,[(n-1)/n, 1].

If we use, right endpoint approximation, then the height of the "k/th" rectangle is \( \left(\frac{k}{n}\right)^2 \).

\[
R_n = \left(\frac{1}{n}\right) \left[ \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{n}{n}\right)^2 \right]
\]

But that summation of the squares of consecutive integers can be shown to to be equivalent to a certain cubic polynomial.
\[
\sum_{k=1}^{n} (k)^2 = \frac{n \cdot (n + 1) \cdot (2 \cdot n + 1)}{6}
\]

\[
R_n = \frac{1}{n^3} \cdot \sum_{k=1}^{n} (k)^2
\]

Substitute the cubic polynomial for the sum.

\[
R_n = \frac{(n + 1) \cdot (2 \cdot n + 1)}{6 \cdot n^2} = \frac{2 \cdot n^2 + 3 \cdot n + 1}{6 \cdot n^2}
\]

Now we use the same approach that we successfully applied to find horizontal asymptotes to find the value that \( R_n \) converges to as we let the number of subintervals, "n", approach " \( \infty \)".
That result, "1/3", is the exact area under the given parabola on the interval \([ 0, 1 ]\). We could have used left endpoints or any points within all of the subintervals and gotten the same exact result!
DEFINITE INTEGRAL
DEFINITION

If "f" is a continuous function over the interval [a, b], we divide [a, b] into "n" subintervals of equal width \( \Delta x \), where

\[
\Delta x = \frac{b - a}{n}
\]

Let: \( x_0, x_1, x_2, \ldots, x_n \) be the endpoints of these subintervals (where, \( x_0 = a \) and \( x_n = b \)) and we choose sample points: \( x_{1s}, x_{2s}, \ldots, x_{ns} \) in these subintervals so \( x_{is} \) lies in the \( i \)th subinterval \([ x_{i-1}, x_i ]\). Then the DEFINITE INTEGRAL of "f" from "a" to "b" is:

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{is}) \cdot \Delta x
\]

The Definite Integral is a number, not a function. It can be positive or negative and still be regarded as an area as the following example illustrates.
DEFINITE INTEGRAL

RIEMANN SUM EXAMPLE #1

\[ f(x) := \frac{1 - x^2}{x^2 + 1} \]

Riemann sum is sum of areas of rectangles above x-axis minus areas below x-axis.

\[ \int_{-2}^{2} f(x) \, dx = A_p + A_n \]
Evaluate Riemann sum for \( f(x) \) taking sample points to be right-hand endpoints with \( n = 6 \) over the interval: \([0, 3]\).

\[
f(x) := x^3 - 6 \cdot x
\]

Here, \( a = 0 \) and \( b = 3 \) with \( n = 6 \).

\[
\Delta x = \left( \frac{b - a}{n} \right) = \frac{3 - 0}{6} = 0.5
\]

So, the righthand endpoints are: \( x_1 = 0.5, x_2 = 1.0, x_3 = 1.5, x_4 = 2.0, x_5 = 2.5, \) and \( x_6 = 3.0 \).

Substitute the values of "\( x_i \)" and find that:

\[
R_6 = -3.937, \text{ not a good approximation!}
\]
DEFINITE INTEGRAL
RIEMANN SUM EXAMPLE # 2 / PART 2

Better estimates for the Definite Integral are obtained by using more rectangles. Assume an arbitrary number, "n", of rectangles.

\[ \Delta x = \frac{b - a}{n} = \frac{3}{n} \]

The right-hand endpoints are: \( x_1 = \frac{3}{n}, \ x_2 = \frac{6}{n} \),

and for the "i/th" endpoint, \( x_i = \frac{3 \cdot i}{n} \).

\[ R_n = \sum_{i=1}^{n} \left[ \left( \frac{3i}{n} \right)^3 - 6 \left( \frac{3i}{n} \right) \right] \cdot \frac{3}{n} \]

\[ R_n = \frac{81}{n^4} \cdot \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \cdot \sum_{i=1}^{n} i \]

How do we evaluate these sums?
FORMULAS FOR CALCULATING SUMS

EVALUATING INTEGRALS: We'll need the following 3 formulas from algebra to evaluate sums with many terms.

\[ \sum_{i=1}^{n} i = \frac{n(n + 1)}{2} \]
\[ \sum_{i=1}^{n} i^2 = \frac{n(n + 1) \cdot (2n + 1)}{6} \]
\[ \sum_{i=1}^{n} i^3 = \left( \frac{n(n + 1)}{2} \right)^2 \]
Replace the sums by their algebraic expressions.

\[ R_n = \frac{81}{n^4} \cdot \sum_{i=1}^{n} i^3 - \frac{54}{n^3} \cdot \sum_{i=1}^{n} i \]

Divide the numerator and denominator by the highest power in the denominator.

\[ R_n = \frac{81 \cdot \left( \frac{n(n + 1)}{2} \right)^2}{n^4} - \frac{54 \cdot n(n + 1)}{n^2} \cdot \frac{2}{2} \]

\[ R_n = \frac{81}{4} \cdot \left( 1 + \frac{1}{n} \right)^2 - 27 \cdot \left( 1 + \frac{1}{n} \right) \]

\[ \int_{0}^{3} f(x) \, dx = \lim_{n \to \infty} R_n = \frac{81}{4} - 27 = -6.75 \]

We get the exact result because we used an infinity of rectangles.
DEFINITE INTEGRAL PROPERTIES

\[
\begin{align*}
\int_{a}^{b} c \, dx &= c \cdot (b - a) \\
\int_{a}^{b} c \cdot f(x) \, dx &= c \cdot \int_{a}^{b} f(x) \, dx \\
\int_{a}^{b} (f(x) + g(x)) \, dx &= \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \\
\int_{a}^{b} (f(x) - g(x)) \, dx &= \int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx
\end{align*}
\]

Finally, if \( a \leq c \leq b \) so that "c" lies within the interval: \([a, b]\), then we have this very important property.

\[
\int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx - \int_{c}^{b} f(x) \, dx
\]
\[ \int_{0}^{3} f(x) \, dx = 4.5 \quad \int_{0}^{4} f(x) \, dx - \int_{3}^{4} f(x) \, dx = 4.5 \]
BLACK SLIDE