tDEC-112
LECTURE # 2

DEFINITE INTEGRAL EVALUATION THEOREM

If "f(x)" is continuous on the interval, [a, b], and "F(x)" is any antiderivative of "f(x)", then:

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]
DEFINITE INTEGRAL EVALUATION PROCESS

• Find an antiderivative, "F(x)".
• Substitute "b" for "x" to get F(b).
• Substitute "a" for "x" to get F(a).
• Subtract F(a) from F(b).

ANTIDERIVATIVES
TABLE / PART 1

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \\
\int e^x \, dx = e^x + C \\
\int \sin(x) \, dx = -\cos(x) + C \\
\int \cos(x) \, dx = \sin(x) + C
\]

\[
\int x^{-1} \, dx = \ln(|x|) + C \\
\int a^x \, dx = \frac{a^x}{\ln(a)} + C
\]
ANTIDERIVATIVES
TABLE / PART 2

\[
\int (\sec(x))^2 \, dx = \tan(x) + C \\
\int (\csc(x))^2 \, dx = -\cot(x) + C \\
\int \sec(x) \cdot \tan(x) \, dx = \sec(x) + C \\
\int \csc(x) \cdot \cot(x) \, dx = -\csc(x) + C \\
\int \frac{1}{x^2 + 1} \, dx = \tan^{-1}(x) + C \\
\int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1}(x) + C
\]

EVALUATION PROCESS
EXAMPLE #1 / PART 1

\[
\int_{-2}^{4} (3 \cdot x - 5) \, dx = -12
\]

\[
f(x) = 3 \cdot x^1 - 5 \cdot x^0 \\
F(x) = \frac{3}{2} \cdot x^2 - \frac{5}{1} \cdot x^1 + C
\]

\[
F(4) = \frac{3}{2} \cdot 4^2 - 5 \cdot 4 + C \\
F(2) = \frac{3}{2} \cdot (-2)^2 - 5 \cdot (-2) + C
\]

\[
F(4) - F(-2) = (4 + C) - (-16 + C) = -12
\]
EVALUATION PROCESS
EXAMPLE # 1 / PART 2

• Graph f(x) & its antiderivative F(x) that satisfies: F(-2) = 0.

“NET” AREA
EXAMPLE # 1 / PART 1

\[ \int_{-1}^{2} x^3 \, dx = 3.75 \]
\[ \int_{-1}^{2} x^3 \, dx = \int_{-1}^{0} x^3 \, dx + \int_{0}^{2} x^3 \, dx \]
\[ \int_{-1}^{0} x^3 \, dx = -0.25, \quad \int_{0}^{2} x^3 \, dx = 4 \]

\[ A_{\text{net}} = -A_{\text{below}} + A_{\text{above}} \]
\[ 3.75 = -0.25 + 4 \]
"NET" AREA
EXAMPLE # 1 / PART 2

The area lies below the x-axis on the interval: [-1, 0)
and above the x-axis on the interval: (0, 2].
The net area is the area above minus the area below.

AREAS OF BOUNDED REGIONS
EXAMPLE # 1 / PART 1

Find the area of the region bounded by x = 0,
y = 0, and x(y), where:

\[ x(y) := 2 \cdot y - y^2 \]
AREAS OF BOUNDED REGIONS

EXAMPLE # 1 / PART 2

The area bounded by this region is "1.333", which can be concluded in three ways:

• Direct application of the "Evaluation Theorem",
\[ \int_{0}^{2} (2 \cdot y - y^2) \, dy = 1.333 \]

• Viewing areas as differences,
\[ 2 - 2 \left[ \int_{0}^{1} 1 - (2 \cdot y - y^2) \, dy \right] = 1.333 \]

• Using your intuitive savvy and knowledge of transformations,
\[ \int_{-1}^{1} 2 \cdot x^2 \, dx = 1.333 \]

FUNDAMENTAL THEOREM OF CALCULUS (FTC)

Suppose "f(x)" is continuous on [a, b],

If \[ g(x) = \int_{a}^{x} f(t) \, dt \] then:

\[ \frac{d}{dx} g(x) = f(x) \]

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]
"FTC"
EVALUATION THEOREM

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]

Substitute "t" for "x"

\[ \int_{a}^{b} f(t) \, dt = F(b) - F(a) \]

Substitute "x" for "b"

\[ \int_{a}^{x} f(t) \, dt = F(x) - F(a) \]

Replace "F(x) - F(a)" with "g(x)".

\[ \int_{a}^{x} f(t) \, dt = g(x) \]

AREA SO FAR FUNCTION

Suppose \( a = 0 \) and \( f(t) = 2t \).

\[ g(x) = \int_{a}^{x} f(t) \, dt \]

Then,

\[ g(x) = \int_{0}^{x} 2 \cdot t \, dt = 2 \left( \frac{x^2}{2} - \frac{0^2}{2} \right) \]

As "x" gets further from zero ( \( a = 0 \)), the area under the graph of "f(x)" increases. So, the integral:

\[ \int_{a}^{x} f(t) \, dt \]

is a definite integral for each value of "x" and is therefore just an extension of the concept of the definite integral.
**PROPERTIES OF INTEGRALS**

\[
\int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx = \int_{a}^{a} f(x) \, dx = 0
\]

\[
\int_{a}^{b} f(x) \, dx \geq 0 \quad \text{if} \quad f(x) \geq 0 \quad \text{on} \ [a, b].
\]

\[
\int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} h(x) \, dx \quad \text{if} \quad f(x) \geq h(x) \quad \text{on} \ [a, b]
\]

\[
m(b - a) \leq \int_{a}^{b} f(x) \, dx \leq M(b - a) \quad \text{if} \quad m \leq f(x) \leq M \quad \text{on} \ [a, b]
\]

**“ROUGH ESTIMATE” PROPERTY APPLICATION**

Estimate the value of the following integral.

\[
\int_{0}^{2} \sqrt{x^3 + 1} \, dx = 3.241
\]

\[
m(b - a) \leq \int_{a}^{b} f(x) \, dx \leq M(b - a) \quad \text{if} \quad m \leq f(x) \leq M \quad \text{on} \ [a, b]
\]

\[
b = 2 \quad a = 0 \quad m = \sqrt{0^3 + 1} = 1 \quad M = \sqrt{2^3 + 1} = 3
\]

\[
2(1) \leq \int_{0}^{2} \sqrt{x^3 + 1} \, dx \leq 2(3)
\]

So, the value of the definite integral lies between "2" and "6", which agrees with the exact result, "3.241".
INTEGRALS & CHAIN RULE

PART 1

Find \( \frac{d}{dx} F(x) \) where \( F(x) = \int_{g(x)}^{h(x)} f(t) \, dt \)

\[
\int_{g(x)}^{h(x)} f(t) \, dt = \int_{g(x)}^{a} f(t) \, dt + \int_{a}^{h(x)} f(t) \, dt
\]

\[
\int_{g(x)}^{a} f(t) \, dt = -\int_{a}^{g(x)} f(t) \, dt
\]

\[
F(x) = -\int_{a}^{g(x)} f(t) \, dt + \int_{a}^{h(x)} f(t) \, dt
\]

PART 2

\[
F(x) = -\int_{a}^{g(x)} f(t) \, dt + \int_{a}^{h(x)} f(t) \, dt
\]

\[
\frac{d}{dx} F(x) = \frac{d}{dg} \left( -\int_{a}^{g(x)} f(t) \, dt \right) \cdot \frac{d}{dx} g(x) + \frac{d}{dh} \left( \int_{a}^{h(x)} f(t) \, dt \right) \cdot \frac{d}{dx} h(x)
\]

\[
\frac{d}{dx} F(x) = -f(g(x)) \cdot \frac{d}{dx} g(x) + f(h(x)) \cdot \frac{d}{dx} h(x)
\]
INTEGRALS & CHAIN RULE

EXAMPLE # 1

Find $F'(x)$ where $F(x)$ is the following integral.

$$F(x) = \int_{\tan(x)}^{x^2} \frac{1}{\sqrt{t^4 + 2}} \, dt$$

$$\frac{d}{dx} F(x) = -f(g(x)) \cdot \frac{d}{dx} g(x) + f(h(x)) \cdot \frac{d}{dx} h(x)$$

$g(x) = \tan(x)$  
$h(x) = x^2$  
$f(t) = \frac{1}{\sqrt{t^4 + 2}}$

$$\frac{d}{dx} g(x) = (\sec(x))^2$$  
$$\frac{d}{dx} h(x) = 2 \cdot x$$

$$\frac{d}{dx} F(x) = \left[ \frac{1}{\sqrt{(x^2)^4 + 2}} \right] \cdot (2 \cdot x) - \left[ \frac{1}{\sqrt{\tan(x)^4 + 2}} \right] \cdot (\sec(x))^2$$

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