DEFINITE INTEGRAL EVALUATION THEOREM

If "f(x)" is continuous on the interval, [a, b], and "F(x)" is any antiderivative of "f(x)", then:

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) \]
DEFINITE INTEGRAL EVALUATION PROCESS

• Find an antiderivative, "F(x)".

• Substitute "b" for "x" to get F(b).

• Substitute "a" for "x" to get F(a).

• Subtract F(a) from F(b).
ANTIDERIVATIVES
TABLE / PART 1

\[
\int x^n \, dx = \frac{x^{n+1}}{n+1} + C
\]

\[
\int e^x \, dx = e^x + C
\]

\[
\int \sin(x) \, dx = -\cos(x) + C
\]

\[
\int \cos(x) \, dx = \sin(x) + C
\]
<table>
<thead>
<tr>
<th>Antiderivative</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int (\sec(x))^2 , dx )</td>
<td>( \tan(x) + C )</td>
</tr>
<tr>
<td>( \int \sec(x) \cdot \tan(x) , dx )</td>
<td>( \sec(x) + C )</td>
</tr>
<tr>
<td>( \int \frac{1}{x^2 + 1} , dx )</td>
<td>( \tan^{-1}(x) + C )</td>
</tr>
<tr>
<td>( \int (\csc(x))^2 , dx )</td>
<td>( -\cot(x) + C )</td>
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<td>( \int \csc(x) \cdot \cot(x) , dx )</td>
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<tr>
<td>( \int \frac{1}{\sqrt{1 - x^2}} , dx )</td>
<td>( \sin^{-1}(x) + C )</td>
</tr>
</tbody>
</table>
**EVALUATION PROCESS**

**EXAMPLE #1 / PART 1**

\[ \int_{-2}^{4} (3 \cdot x - 5) \, dx = -12 \]

\[
\begin{align*}
    f(x) &= 3 \cdot x^1 - 5 \cdot x^0 \\
    F(x) &= \frac{3}{2} \cdot x^2 - \frac{5}{1} \cdot x^1 + C
\end{align*}
\]

\[
\begin{align*}
    F(4) &= \frac{3}{2} \cdot 4^2 - 5 \cdot 4 + C \\
    F(2) &= \frac{3}{2} \cdot (-2)^2 - 5 \cdot (-2) + C
\end{align*}
\]

\[
F(4) - F(-2) = (4 + C) - (-16 + C) = -12
\]
• Graph \( f(x) \) & its antiderivative \( F(x) \) that satisfies: \( F(-2) = 0 \).
“NET” AREA
EXAMPLE # 1 / PART 1

\[
\int_{-1}^{2} x^3 \, dx = 3.75
\]

\[
\int_{-1}^{2} x^3 \, dx = \int_{-1}^{0} x^3 \, dx + \int_{0}^{2} x^3 \, dx
\]

\[
\int_{-1}^{0} x^3 \, dx = -0.25
\]

\[
\int_{0}^{2} x^3 \, dx = 4
\]

\[
A_{\text{net}} = -A_{\text{below}} + A_{\text{above}}
\]

\[
3.75 = -0.25 + 4
\]
The area lies below the x-axis on the interval: [ -1, 0) and above the x-axis on the interval: (0, 2]. The net area is the area above minus the area below.
AREAS OF BOUNDED REGIONS
EXAMPLE # 1 / PART 1

Find the area of the region bounded by \( x = 0, \ y = 0, \) and \( x(y), \) where:

\[
x(y) := 2 \cdot y - y^2
\]
The area bounded by this region is "1.333", which can be concluded in three ways:

- Direct application of the "Evaluation Theorem",
  \[ \int_{0}^{2} \left( 2 \cdot y - y^2 \right) \, dy = 1.333 \]

- Viewing areas as differences,
  \[ 2 - 2 \left[ \int_{0}^{1} 1 - \left( 2 \cdot y - y^2 \right) \, dy \right] = 1.333 \]

- Using your intuitive savvy and knowledge of transformations,
  \[ \int_{-1}^{1} 2 \cdot x^2 \, dx = 1.333 \]
FUNDAMENTAL THEOREM OF CALCULUS (FTC)

Suppose "f(x)" is continuous on [a, b],

If \( g(x) = \int_{a}^{x} f(t) \, dt \) then:

\[
\frac{d}{dx} g(x) = f(x)
\]

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]
“FTC”
EVALUATION THEOREM

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]

Substitute "t" for "x"
\[
\int_{a}^{b} f(t) \, dt = F(b) - F(a)
\]

Substitute "x" for "b".
\[
\int_{a}^{x} f(t) \, dt = F(x) - F(a)
\]

Replace "F(x) - F(a)" with "g(x)".
\[
\int_{a}^{x} f(t) \, dt = g(x)
\]
Suppose $a = 0$ and $f(t) = 2t$. Then,

$$g(x) = \int_a^x f(t) \, dt$$

Then,

$$g(x) = \int_0^x 2 \cdot t \, dt = 2 \left( \frac{x^2}{2} - \frac{0^2}{2} \right)$$

As "x" gets further from zero ($a = 0$), the area under the graph of "f(x)" increases. So, the integral:

$$\int_a^x f(t) \, dt$$

is a definite integral for each value of "x" and is therefore just an extension of the concept of the definite integral.
PROPERTIES OF INTEGRALS

\[
\int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx 
\]

\[
\int_{a}^{a} f(x) \, dx = 0
\]

\[
\int_{a}^{b} f(x) \, dx \geq 0 \quad \text{if} \quad f(x) \geq 0 \quad \text{on} \quad [a, b].
\]

\[
\int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} h(x) \, dx \quad \text{if} \quad f(x) \geq h(x) \quad \text{on} \quad [a, b]
\]

\[
m(b - a) \leq \int_{a}^{b} f(x) \, dx \leq M(b - a) \quad \text{if} \quad m \leq f(x) \leq M \quad \text{on} \quad [a, b]
\]
“ROUGHESTIMATE” PROPERTY APPLICATION

Estimate the value of the following integral.

\[
\int_{0}^{2} \sqrt{x^3 + 1} \, dx = 3.241
\]

\[
m(b - a) \leq \int_{a}^{b} f(x) \, dx \leq M(b - a) \quad \text{if} \quad m \leq f(x) \leq M \quad \text{on} \quad [a, b]
\]

\[
b = 2 \quad a = 0 \quad m = \sqrt{0^3 + 1} = 1 \quad M = \sqrt{2^3 + 1} = 3
\]

\[
2(1) \leq \int_{0}^{2} \sqrt{x^3 + 1} \, dx \leq 2(3)
\]

So, the value of the definite integral lies between "2" and "6", which agrees with the exact result, "3.241".
Find \( \frac{d}{dx} F(x) \) where \( F(x) = \int_{a}^{h(x)} f(t) \, dt \)
INTEGRALS & CHAIN RULE

PART 2

\[ F(x) = -\int_a^g f(t) \, dt + \int_a^h f(t) \, dt \]

\[ \frac{d}{dx} F(x) = \frac{d}{dg} \left( -\int_a^g f(t) \, dt \right) \cdot \frac{d}{dx} g + \frac{d}{dh} \left( \int_a^h f(t) \, dt \right) \cdot \frac{d}{dx} h \]

\[ \frac{d}{dx} F(x) = -f(g(x)) \cdot \frac{d}{dx} g(x) + f(h(x)) \cdot \frac{d}{dx} h(x) \]
EXAMPLE # 1

Find \( F'(x) \) where \( F(x) \) is the following integral.

\[
F(x) = \int_{\tan(x)}^{x^2} \frac{1}{\sqrt{t^4 + 2}} \, dt
\]

\[
\frac{d}{dx} F(x) = -f(g(x)) \cdot \frac{d}{dx} g(x) + f(h(x)) \cdot \frac{d}{dx} h(x)
\]

\( g(x) = \tan(x) \quad h(x) = x^2 \quad f(t) = \frac{1}{\sqrt{t^4 + 2}} \)

\[
\frac{d}{dx} g(x) = (\sec(x))^2 \quad \frac{d}{dx} h(x) = 2 \cdot x
\]

\[
\frac{d}{dx} F(x) = \left[ \frac{1}{\sqrt{(x^2)^4 + 2}} \right] \cdot (2 \cdot x) - \left( \frac{1}{\sqrt{\tan(x)^4 + 2}} \right) \cdot (\sec(x))^2
\]
BLACK SLIDE